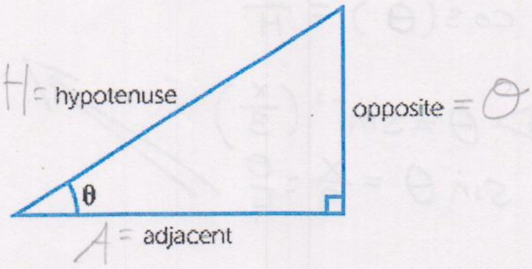
Math 229: Supplement on composition of inverses. (Section 8.3)

Write all 6 trig functions based on the Right Triangle Definitions

$\sin\theta = \frac{O}{H}$ $\csc\theta = \frac{H}{O}$

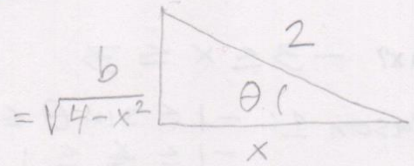
$\cos\theta = \frac{A}{H}$ $\sec\theta = \frac{H}{A}$

$\tan\theta = \frac{O}{A}$ $\cot\theta = \frac{A}{O}$



Find a formula for each of the following compositions by constructing a right triangle.

1. $\sin\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$ Right triangle:



$x^2 + b^2 = 2^2$

$x^2 + b^2 = 4$

$b^2 = 4 - x^2$

$b = \pm\sqrt{4-x^2}$

We'll discard the negative root since we're dealing with an acute angle (QI) See note (*) below!

$= \sin(\theta) = \frac{O}{H}$

$\theta = \cos^{-1}\left(\frac{x}{2}\right)$

$\cos\theta = \frac{x}{2} = \frac{A}{H}$

$\sin\left(\cos^{-1}\left(\frac{x}{2}\right)\right) = \frac{O}{H} = \frac{\sqrt{4-x^2}}{2}$

a. What is the restriction on x? $-2 \le x \le 2$

Why is this restriction necessary?

Reason 1: The radicand must be positive for the root to be a real number!

Reason 2:

$-1 \le \cos\theta \le 1$

$-1 \le \frac{x}{2} \le 1$

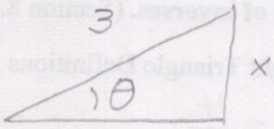
$-2 \le x \le 2$

b. Check your formula by using x = 0, 2

$\sin\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$	$\stackrel{?}{=} \frac{\sqrt{4-x^2}}{2}$
$x=0$: $\sin(\cos^{-1}(0))$	$\frac{\sqrt{4-0^2}}{2}$
$\sin\left(\frac{\pi}{2}\right)$	$= \frac{2}{2}$
1	$= 1$
$x=2$: $\sin(\cos^{-1}(1))$	$= \frac{\sqrt{4-2^2}}{2}$
$\sin(0)$	$= \frac{\sqrt{0}}{2}$
0	$= 0$

(*) Note: since the range of $\cos^{-1}\left(\frac{x}{2}\right)$ is restricted to QI, QII, and the axes (i.e. $0 \le \cos^{-1}\left(\frac{x}{2}\right) \le \pi$) and we know that $\sin\theta$ is positive in QI and QII, it must be that $\sin\left(\cos^{-1}\left(\frac{x}{2}\right)\right)$ is positive (or 0 or 1 for the axis angles.)

2. $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ Right triangle:



$a = \sqrt{9-x^2}$

$a^2 + x^2 = 3^2$

$a^2 = 9 - x^2$

$a = \pm\sqrt{9-x^2}$

Again, we'll take only the positive square root. (*)

$= \cos(\theta) = \frac{A}{H}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\sin\theta = \frac{x}{3} = \frac{O}{H}$

$\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \frac{A}{H} = \frac{\sqrt{9-x^2}}{3}$

a. What is the restriction on x? $-3 \le x \le 3$

Why is this restriction necessary?

Reason 1: $-1 \le \sin\theta \le 1$

$-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$

Reason 2: $9 - x^2 \ge 0$

By inspection: $-3 \le x \le 3$

b. Check your formula by using x = -3, 0

$\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) \stackrel{?}{=} \frac{\sqrt{9-x^2}}{3}$

$x = -3$ $\quad \cos(\sin^{-1}(-1)) \stackrel{?}{=} \frac{\sqrt{9-(-3)^2}}{3}$

$\cos\left(-\frac{\pi}{2}\right) \stackrel{?}{=} \frac{\sqrt{0}}{3}$

$0 = 0$ ✓

$x = 0$ $\quad \cos(\sin^{-1}(0)) \stackrel{?}{=} \frac{\sqrt{9-0^2}}{3}$

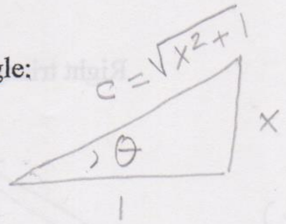
$\cos(0) \stackrel{?}{=} \frac{3}{3}$

$1 = 1$ ✓

(*) Since $\sin^{-1}\left(\frac{x}{3}\right)$ is restricted to QI and QIV $\left(-\frac{\pi}{2} \le \sin^{-1}\left(\frac{x}{3}\right) \le \frac{\pi}{2}\right)$ and $\cos(\theta)$ is positive in QI and QIV, it must be that $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ is positive (or 1 or 0 on the axes).

3. $\sec\left(\tan^{-1}(x)\right)$

Right triangle:



$x^2+1^2=c^2$

$x^2+1=c^2$

$c^2=x^2+1$

$c=\pm\sqrt{x^2+1}$

Again, we can discard the negative for an acute angle (See note (*) below)

$=\sec(\theta)$

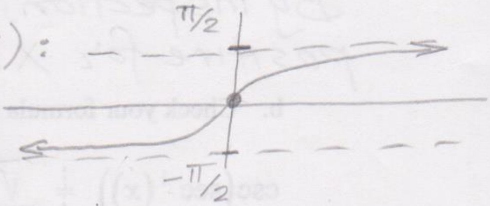
$\theta=\tan^{-1}(x)$

$\tan\theta=\frac{x}{1}=\frac{O}{A}$

$\sec\left(\tan^{-1}(x)\right)=\frac{H}{A}=\frac{\sqrt{x^2+1}}{1}=\sqrt{x^2+1}$

a. What is the restriction on x? There is no restriction! $(-\infty,\infty)$
Why? The domain of $\tan^{-1}(x)$ is all real numbers!

Graph of $\tan^{-1}(x)$:



b. Check your formula by using x = 0

$\sec\left(\tan^{-1}(x)\right) \stackrel{?}{=} \sqrt{x^2+1}$

$x=0$ $\quad\sec\left(\tan^{-1}(0)\right)=\sqrt{0^2+1}$

$\sec(0)=\sqrt{1}$

$1=1$ ✓

Also, x^2+1 is always positive so $\sqrt{x^2+1}$ is real for all x-values!

(*) Note:
The same analysis of the sign on $\pm\sqrt{x^2+1}$ applies here.
The range of $\tan^{-1}(x)$ is QI and QIV, and $\sec\theta$ is positive for all angles in QI and QIV.

Notes:

$\theta=\tan^{-1}(0)$

$\tan\theta=0$

$\theta=0$

$\sec 0=\frac{1}{\cos 0}$

$=\frac{1}{1}$

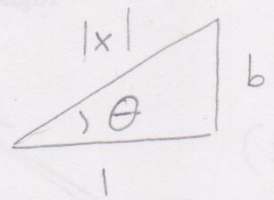
$=1$

4. $\csc\left(\sec^{-1}(x)\right)$ Right triangle:

$= \csc(\theta)$

$\Rightarrow \theta = \sec^{-1}(x)$

$\sec\theta = \frac{x}{1} = \frac{H}{A}$



Wait! Why is the hypotenuse the absolute value of x?

The r-value in the coordinate definition of trig functions must be positive, i.e., the hypotenuse must be positive.

So apply the absolute value, and voila! Issue resolved!

$\csc\left(\sec^{-1}(x)\right) = \frac{H}{O} = \frac{|x|}{\sqrt{x^2-1}}$

a. What is the restriction on x? $(-\infty,-1] \cup [1,\infty)$

By inspection, x^2-1 will be positive for $x \geq 1$ OR $x \leq -1$

b. Check your formula by using x = 2

$\csc\left(\sec^{-1}(x)\right) \doteq \frac{|x|}{\sqrt{x^2-1}}$

$x=2$ $\quad \csc(\sec^{-1}(2)) \doteq \frac{|2|}{\sqrt{2^2-1}}$

$\csc\left(\frac{\pi}{3}\right)$

$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$

Check for $x=-2$
(to illustrate)

$\csc(\sec^{-1}(-2)) \overset{?}{=} \frac{|-2|}{\sqrt{(-2)^2-1}}$

$\csc\left(\frac{2\pi}{3}\right) = \frac{2}{\sqrt{3}}$

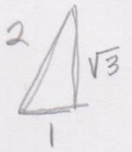
$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}$

Note:

$\sec^{-1}(2) = \theta$

$\frac{2}{1} = \sec\theta$

$\theta = \frac{\pi}{3}$



$\sec^{-1}(-2) = \theta$

$-\frac{2}{1} = \sec\theta$

QII

$\theta = \frac{2\pi}{3}$