Math 229: Test 3 **Name:** KEY

(100 points)

Please do your work in a well-organized manner. Credit is based on the amount of correct work shown, not just on the final answer. Use proper notation. Only *scientific calculators* are allowed on the exam.

1. (4 pts) Evaluate without using a calculator. Write your answer in RADIANS.

(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

(b) $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(output from arctan(x) must be between $-\frac{\pi}{2}$ and $\pi/2$.

2. (2 pts) Use your calculator to evaluate. Give your answer in DEGREES, to the nearest 10th of a degree.

(a) $\arccos(-.9938) = 173.6°$

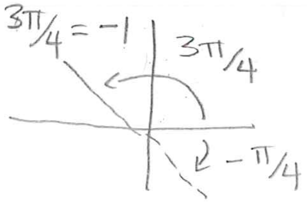
(b) $\tan^{-1}(1000) = 89.9°$

3. (4 pts) Explain why $\sin^{-1}\left(\frac{3}{2}\right)$ is undefined. Answers may vary

If $\theta = \sin^{-1}\left(\frac{3}{2}\right)$
then $\sin\theta = \frac{3}{2}$ but it's impossible for the side of a right triangle to be greater than the hypotenuse.

4. (3 pts) Evaluate without using a calculator. $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right) = \boxed{-\frac{\pi}{4}}$

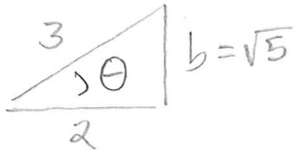
$\tan 3\pi/4 = -1$



5. (6 pts) Evaluate without using a calculator. Write your answer using exact values. Include a sketch!

$\sin\left(\cos^{-1}\left(\frac{2}{3}\right)\right) = \sin\theta$ where $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

$= \boxed{\frac{\sqrt{5}}{3}}$

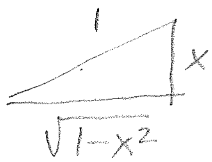


$2^2 + b^2 = 3^2$
$b^2 = 5$
$b = \sqrt{5}$

6. (6 pts) Find an algebraic expression in terms of x for $\tan\left(\sin^{-1}(x)\right)$. Include a sketch!

$\theta = \sin^{-1} \frac{x}{1}$

$\sin\theta = \frac{x}{1}$



$= \boxed{\frac{x}{\sqrt{1-x^2}}}$

7. (3 pts) Prove tangent is an odd function by showing that $\tan(-x) = -\tan(x)$

$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$ Q.E.D.

8. (3 pts) Prove that $\sin(-x)\sec(-x) = -\tan(x)$

$\sin(-x)\sec(-x)$

$= -\sin(x) \cdot \frac{1}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$ $\blacksquare$

Helpful identities:

$\sin(A+B) = \sin A \cos B + \sin B \cos A$

$\cos(A+B) = \cos A \cos B - \sin B \sin A$

$1 + \tan^2 A = \sec^2 A$

$\cot^2 A + 1 = \csc^2 A$

$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$

$\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$

9. (3 pts) Write the expression as a single trigonometric function:

$\sin(4x)\cos(6x) + \sin(6x)\cos(4x) =$ $\sin(4x+6x) = \sin(10x)$

10. (5 pts) Write the following identities from memory: $\cos^2\theta + \sin^2\theta = 1$

(a) Pythagorean identity for sine and cosine OR $\sin^2\theta + \cos^2\theta = 1$

(b) Double angle identity for sine $\sin(2\theta) = 2\sin\theta\cos\theta$

(c) Double angle identity for cosine (all three forms) $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta$

11. (5 pts) Use the Sum of Angles Identity to derive the Double Angle Identity for sine.

$$\sin(2\theta) = \sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$$
$$= 2\sin\theta\cos\theta$$

12. (5 pts) Use the Pythagorean Identity for sine, cosine to derive the Pythagorean Identity for tangent, secant

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \longrightarrow 1 + \tan^2\theta = \sec^2\theta$$

13. (6 pts) Prove that $\dfrac{1}{1-\cos A} + \dfrac{1}{1+\cos A} = 2\csc^2 A$

$$\frac{1}{(1-\cos A)}\frac{(1+\cos A)}{(1+\cos A)} + \frac{1}{(1+\cos A)}\frac{(1-\cos A)}{(1-\cos A)}$$

$$= \frac{1+\cos A}{1-\cos^2 A} + \frac{1-\cos A}{1-\cos^2 A}$$

$$= \frac{1+\cos A + 1-\cos A}{1-\cos^2 A} \longrightarrow = \frac{2}{\sin^2 A}$$

$$= 2\csc^2 A \quad \blacksquare$$

14. (6 pts) Prove that $\cos^4 x - \sin^4 x = \cos(2x)$

$$\cos^4 x - \sin^4 x$$

$$= (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$$

$$= (\cos^2 x - \sin^2 x)(1)$$

$$= \cos(2x) \quad \blacksquare$$

15. (5 pts) Prove that $\sin\frac{A}{2}\cos\frac{A}{2} = \frac{1}{2}\sin A$

XC

$$\sin\frac{A}{2}\cdot\cos\frac{A}{2}$$

$$= \sqrt{\frac{1-\cos A}{2}}^{\,*}\cdot\sqrt{\frac{1+\cos A}{2}}^{\,*}$$

$$= \sqrt{\frac{1-\cos^2 A}{4}}$$

$$\rightarrow = \sqrt{\frac{\sin^2 A}{4}}$$

$$= \frac{\sin A}{2}$$

$$= \frac{1}{2}\sin A$$

note: we have to have the restriction that $\sin A \geq 0$ so $0 \leq A \leq \pi$ since $\sqrt{\ }$'s can't have negative output values

XC $*$: Restriction: $\frac{A}{2} \in QI$

16. (10 pts) Find all solutions (the general solution) in $\boxed{\text{degrees:}}$

$\sqrt{2}\cos\theta + 1 = 0$

$$\cos\theta = -\frac{1}{\sqrt{2}}$$

$$\theta_{REF} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$$

QII

$$180^\circ - 45^\circ = 135^\circ$$

$$\boxed{\theta = 135^\circ + 360^\circ k}$$

QIII

$$180^\circ + 45^\circ = 225^\circ$$

$$\boxed{\theta = 225^\circ + 360^\circ k}$$

17. (10 pts) Find all solutions (the general solution) in $\boxed{\text{radians:}}$

$\sin(2x) - \cos x = 0$

$$2\sin x\cos x - \cos x = 0$$

$$\cos x\,(2\sin x - 1) = 0$$

$$\cos x = 0$$

$$\boxed{x = \frac{\pi}{2} + n\pi} \quad \text{Best}$$

OR $\begin{cases} x = \frac{\pi}{2} + 2n\pi \\ x = \frac{3\pi}{2} + 2n\pi \end{cases}$ ok

$$2\sin x - 1 = 0$$

$$\sin x = \frac{1}{2}$$

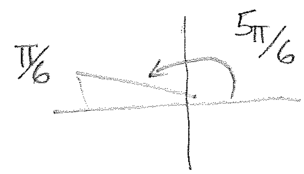
$$x_{Ref} = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

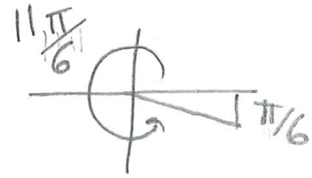
QI

$$\boxed{\theta = \frac{\pi}{6} + 2\pi k}$$

QII

$$\boxed{\theta = \frac{5\pi}{6} + 2\pi k}$$





18. (10 pts) Find all solutions (the general solution) in radians:

$\cos(3x) = \frac{\sqrt{3}}{2}$

$\{\theta = 3x\}$ $\cos\theta = \frac{\sqrt{3}}{2}$

$\theta_{Ref} = \cos^{-1}\frac{\sqrt{3}}{2}$

$= \frac{\pi}{6}$

QI	QIV
$\theta = \frac{\pi}{6} + 2\pi k$	$\theta = \frac{11\pi}{6} + 2\pi k$
$\frac{3x}{3} = \frac{\frac{\pi}{6} + 2\pi k}{3}$	$3x = \frac{11\pi}{6} + 2\pi k$
$\boxed{x = \frac{\pi}{18} + \frac{2\pi k}{3}}$	$\boxed{x = \frac{11\pi}{18} + \frac{2\pi k}{3}}$

19. (6 pts) (a) What are the two equations (y =) that you would need to graph in order to solve $\tan(2x) = \sqrt{3}$

Equation 1: $y = \tan(2x)$

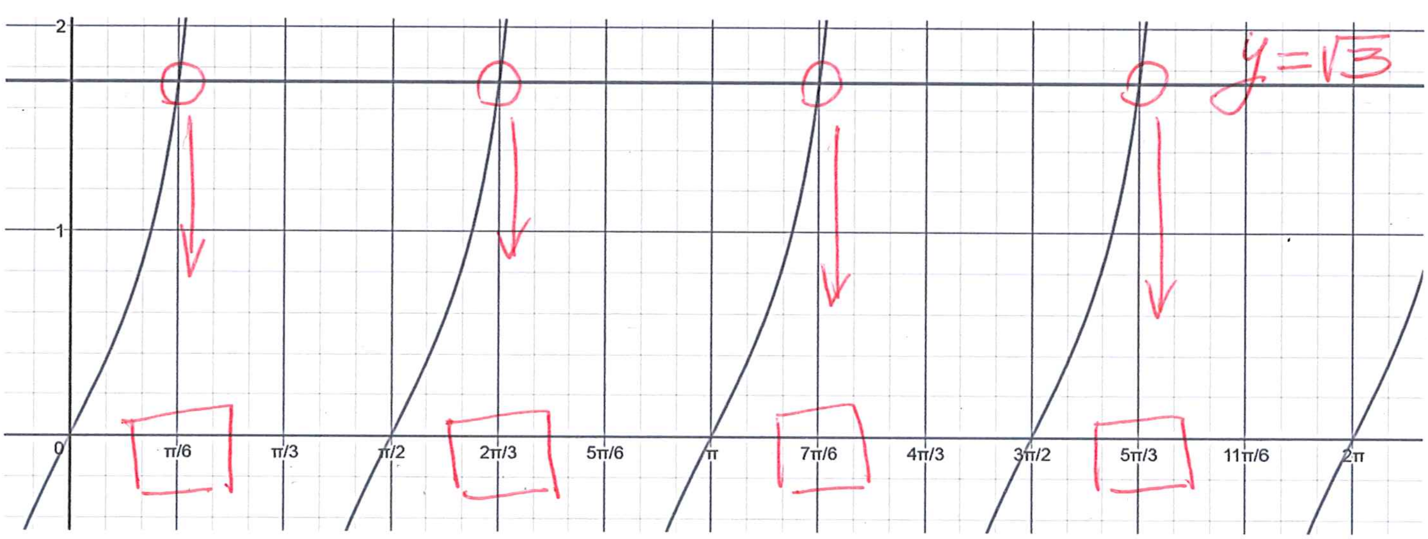
Equation 2: $y = \sqrt{3}$

(b) The two equations are graphed below. Use the graph to identify the solutions to $\tan(2x) = \sqrt{3}$ for $0 \le x \le 2\pi$.

Solutions: $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

(c) Extra credit (2 pts) : From your answer above, write a formula for ALL of the solutions to this equation (in radians). No credit for solving the equation algebraically, only if for finding the pattern from your solutions listed in part (b)

General solution: $x = \frac{\pi}{6} + \frac{\pi}{2}k$



Extra credit (6 points):

The equation that gives the height, h, (in feet) of a passenger on a particular Ferris wheel at any time t (in minutes) is $h = 160 - 150\cos\left(\frac{\pi}{20}t\right)$ $B = \frac{\pi}{20}$ $T = \frac{2\pi}{B} = \frac{2\pi}{\pi/20} = 40$ minutes! Yikes!

(a) What is the highest point the passenger will reach and when will she reach that point? (You may solve "by inspection" if you can, i.e., you don't have to show work.)

High point: when $\cos\left(\frac{\pi}{20}t\right) = -1$ so $t = 20$ minutes!

$h = 160 + 150 = 310$ feet

and it's SUPER tall, also!

This is one slooow Ferris Wheel!

(b) What is the lowest point the passenger will reach and when will she reach that point? (You may solve "by inspection" if you can, i.e., you don't have to show work.)

Lowest point: when $\cos\left(\frac{\pi}{20}t\right) = 1$ so $t = 0$ minutes

$h = 160 - 150 = 10$ feet

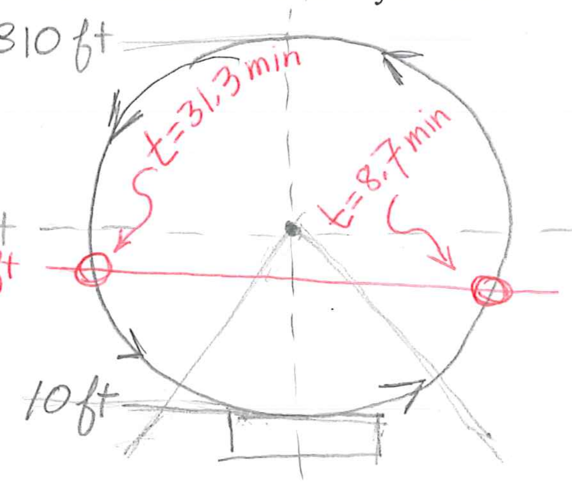
(makes sense – it's at its low point when the clock starts.

(c) Find all the times when the passenger will be 130 feet off the ground. Round your answer to the nearest tenth of a minute. SHOW WORK. Draw a diagram of the Ferris wheel to illustrate the solution you found.

$130 = 160 - 150\cos\left(\frac{\pi}{20}t\right)$

$\frac{-30}{-150} = \frac{-150\cos\left(\frac{\pi}{20}t\right)}{-150}$

$.2 = \cos\left(\frac{\pi}{20}t\right)$



QI	QIV
$\frac{\pi}{20}t = \cos^{-1}(.2)$	$\frac{\pi}{20}t = 2\pi - \cos^{-1}(.2)$
$t = \frac{20}{\pi}\cos^{-1}(.2)$	$t = 40 - \frac{20}{\pi}\cos^{-1}(.2)$
$t = 8.7$ minutes	$t = 31.3$ minutes

Radius = 150 ft!

to get ALL times add 40k ⟹ All times:

$t = 8.7 + 40k$ minutes $k = 0, 1, 2, \ldots$

$t = 31.3 + 40k$ minutes $k = 0, 1, 2, \ldots$