Math 265A: Test 3 **Name and Date:** KEY
Shortcuts to Differentiation

Only scientific, non-graphing calculators may be used for this exam. Please show all of your work in neat, clear steps. Credit is based on the amount of correct work shown, not just the final answer.

1. (5 pts) Write out the FORMAL defintion of $f'(x)$.

If f is a ~~continuous function for all x~~

then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

provided the limit exists.

2. (5 pts) Use the limit definition of the derivative to find a formula for $f'(x)$, given $f(x) = x^2 + 3x$
Make sure your work is very clear and your notation is correct.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h}$$
$$= \lim_{h \to 0} 2x + h + 3$$
$$= \boxed{2x + 3}$$

3. (6 pts) Fill in the following rules of differentiation. c, p and a are real-valued constants, a > 0.

$\frac{d}{dx}[x^p] = p \cdot x^{p-1}$

$\frac{d}{dx}[a^x] = \ln a \cdot a^x$

$\frac{d}{dx}[\ln x] = \frac{1}{x}$

$\frac{d}{dx}[\cos x] = -\sin x$

$\frac{d}{dx}[\sin x] = \cos x$

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}[\arctan x] =$ ~~[illegible]~~ $\frac{1}{x^2+1}$
nope!

$\frac{d}{dx}[cf(x)] = c \cdot f'(x)$

$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

$\frac{d}{dx}[F(x)S(x)] = F'(x)S(x) + F(x)S'(x)$

$\frac{d}{dx}\left[\frac{T(x)}{B(x)}\right] = \frac{T'(x)B(x) - T(x)B'(x)}{[B(x)]^2}$

$\frac{d}{dx}[f(g(x))]$ $f'(g(x)) \cdot g'(x)$

4. (32 pts) Find the derivative for each function. **Make sure to use the correct derivative notation** for each of the problems. You do NOT have to simplify your answers.

a. $y = \frac{x^2+1}{x^2-1}$ Quotient Rule

$$\frac{dy}{dx} = \frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}$$ ok

$$= \frac{-4x}{(x^2-1)^2}$$

b. $F = (\ln(m))^2 - \ln(m^2)$ Chain Rule

$$\frac{dF}{dm} = 2(\ln(m))^1 \frac{1}{m} - \frac{1}{m^2} \cdot 2m$$ ok

$$= \frac{2\ln(m) - 2}{m}$$

c. $g(t) = 4e^{(3t^2+7t)}$ Chain Rule

$$g'(t) = 4e^{3t^2+7t} \cdot (6t+7)$$ ok

$$= 4(6t+7)e^{3t^2+7t}$$

d. $f(x) = e^{-x}\sin(x)$ Product Rule

$$f'(x) = e^{-x}(-1)\sin x + e^{-x} \cdot \cos x$$ ok

$$= e^{-x}(-\sin x + \cos x)$$

e. $s(x) = \arctan(x-2)$

$$s'(x) = \frac{1}{(x-2)^2+1}$$

f. $h(x) = x^\pi + \pi^x$

$$h'(x) = \pi x^{\pi-1} + \ln\pi \cdot \pi^x$$

g. $f(t) = \cos^2(3t+5)$ Chain Rule

$$= (\cos(3t+5))^2$$

ok

$$f'(t) = 2\cos(3t+5) \cdot (-\sin(3t+5)) \cdot 3$$

$$= -6\cos(3t+5)\sin(3t+5)$$

h. $f(y) = \sqrt[3]{y} - \frac{3}{y^2} + 5$ $= y^{\frac{1}{3}} - 3y^{-2} + 5$

$$f'(y) = \frac{1}{3}y^{-\frac{2}{3}} + 6y^{-3}$$ ok

$$= \frac{1}{3y^{2/3}} + \frac{6}{y^3}$$

Product Rule: 3pts

5. (15 pts) 5 (a) Given that $2x^2y+3y^2=5$, find $\frac{dy}{dx}$

$$\frac{d}{dx}\left(2x^2 \cdot y+3y^2\right)=\frac{d}{dx}(5)$$

$$4xy+2x^2\frac{dy}{dx}+6y\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left(2x^2+6y\right)=-4xy \Rightarrow \boxed{\frac{dy}{dx}=\frac{-4xy}{2x^2+6y}}$$

better $\frac{dy}{dx}=\frac{-2xy}{x^2+3y}$

3 (b) Determine the slope of the line tangent to the curve given by the equation above, at the point (-1, 1)

$$m_{tan}\Big|_{(-1,1)}=\frac{dy}{dx}\Big|_{(-1,1)}=\frac{-4(-1)(1)}{2(-1)^2+6(1)}=\frac{4}{8}=\boxed{\frac{1}{2}}$$

4 (c) Referring to part (a), for what values of x and/or y is the tangent horizontal?

tangent line is horizontal when $m_{tan}=0$

$x=0$ is a valid solution since $2(0)^2y+3y^2=5 \Rightarrow y=\pm\sqrt{\frac{5}{3}}$; $y=0$ is NOT valid $2x^2(0)+3(0)^2=5 \Rightarrow 0=5 \Rightarrow$ no solution

$$\frac{dy}{dx}=\frac{-4xy}{2x^2+6y}=0 \text{ for } \boxed{x=0 \text{ or } \cancel{y=0}}$$

Note: Exclude $x=0$ AND $y=0$

6. (4 pts) The table below gives values for functions f and g and their derivatives. Use it to find the following:

x	0	2	3
$f(x)$	3	0	1
$g(x)$	2	3	5
$f'(x)$	-2	-1	6
$g'(x)$	3	2.5	4

a. $\frac{d}{dx}(f(x)g(x))$ at x = 0

$$=f'(0)\cdot g(0)+f(0)\cdot g'(0)$$
$$=-2(2)+(3)(3)$$
$$=\underline{\underline{5}}$$

b. $\frac{d}{dx}(f(g(x)))$ at x = 2

$$=f'(g(2))\cdot g'(2)$$
$$=f'(3)\cdot(2.5)$$
$$=6(2.5)=\underline{\underline{15}}$$

7. (4 pts) Prove: $\frac{d}{dx}(\ln x) = \frac{1}{x}$ Include a justification for each step.

(Hint: Begin with $e^{\ln(x)} = x$.) $e^{\ln x} = x$ Inverse relationship of e^x, $\ln x$

$\frac{d}{dx}\left[e^{\ln(x)}\right] = \frac{d}{dx}[x]$ Differentiation preserves equality

$e^{\ln(x)} \cdot \frac{d}{dx}[\ln(x)] = 1$ Chain Rule; Power Rule

$x \cdot \frac{d}{dx}[\ln(x)] = 1$ Inverse relationship of e^x; $\ln x$.

$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ Q.E.D. (Multiplication) Division property of equality.

$x \neq 0$

8. (15 pts) Given $f(x) = x^3$

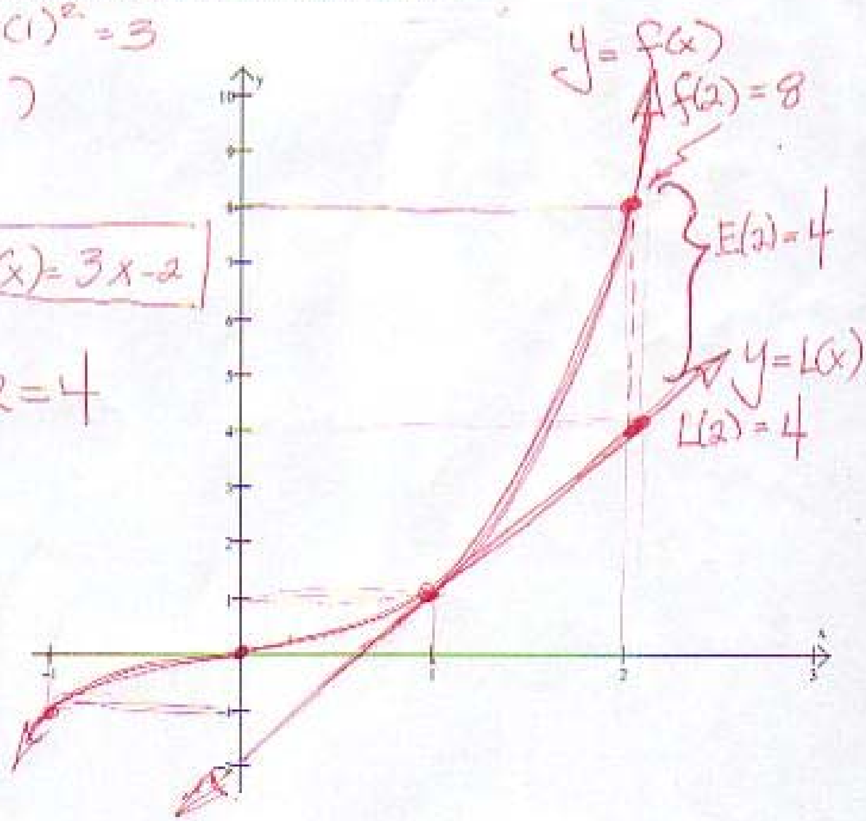
6pts (a) find the "local linearization," $L(x)$ (equation of the tangent line) at the point x = 1.

$f'(x) = 3x^2$, $m_{tan} = f'(+1) = 3(1)^2 = 3$

$(x_1, y_1) = (1, f(1)) = (1, 1)$

$y - 1 = 3(x - 1)$

$\boxed{y = 3x - 2}$ or $\boxed{L(x) = 3x - 2}$



3 pts (b) Estimate the value of $f(2)$ using $L(x)$

$f(2) \doteq L(2) = 3(2) - 2 = 4$

3 pts (c) Find the Error of Estimation for your result in part (b)

Error: $E(2) = f(2) - L(2)$
$= (2)^3 - 4$
$= 8 - 4 = 4$

(d) Sketch a careful, detailed graph that illustrates the f, L, and the Error value from parts (a), (b) and (c)

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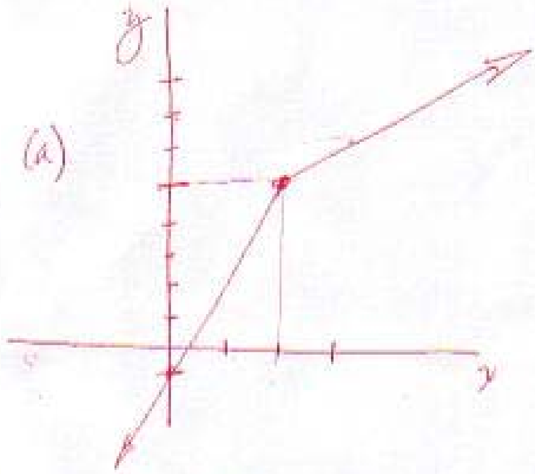
9. (8 pts) Using the given function, do the following:
(a) Graph it
(b) Identify the point at which f is not differentiable.
(c) Explain why f is not differentiable at this point.

$$f(x) = \begin{cases} 3x-1 & x<2 \\ x+3 & x \geq 2 \end{cases}$$

(a)



(b) f is not differentiable at x = 2

(c) There is an abrupt change in the slope of the graph at x = 2
So $m_{tan} = 3$ for $x < 2$ and $m_{tan} = 1$ for $x > 2$

10. (6 pts) True or false:

(a) If a function is continuous at a point, it is also differentiable at that point. false

(b) There are functions which are differentiable for all x but not continuous for all x false

(c) If a function is differentiable at a point x = a, then $f(a)$ must exist. true