Math 265A: Test 4, Part I Name: KEY

Applications of the Derivative

Please organize all of your work in a clear, logical manner. Credit for each problem is based on the amount of correct work shown, not just the final answer. Graphing calculators are permitted on this exam.

1. Give all the critical points of the following functions:

5 (a) $g(t) = t + \cos(t) \qquad 0 \le t \le 2\pi$

$g'(t) = 1 - \sin(t)$

$g'(t) = 0 \Rightarrow 1 - \sin(t) = 0$

$\Rightarrow \sin(t) = 1$

$\Rightarrow t = \pi/2$

Critical point(s): $t = \pi/2$

5 (b) $h(x) = kx^{\frac{2}{3}}$, where k is a real-valued constant

$h'(x) = \frac{2}{3} k x^{-\frac{1}{3}} = \frac{2k}{3x^{1/3}}$

$h'(x) = 0 \Rightarrow \frac{2k}{3x^{1/3}} = 0$ (no solution)

$h'(x)$ is undefined when $x = 0$

Critical point(s): $x = 0$

2. Given the function $f(x) = e^{-3x^2}$

7 (a) Determine the global max and min of f on the interval [-2 , 1]

$f'(x) = -6xe^{-3x^2}$

$f'(x) = 0 \Rightarrow -6xe^{-3x^2} = 0$

$\Rightarrow -6x = 0, \; x = 0$

Note: $e^{-3x^2} \neq 0$ for any x

x	y
-2	e^{-12}
0	$e^0 = 1$
1	e^{-1}

The global min is e^{-12} (found at the endpoint, $x = -2$)

The global max is 1 (found at $x = 0$)

12 (b) Determine the interval(s) on which f is concave up and concave down. Show work that supports your answer. Also give the coordinates of any points of inflection. Use exact values when giving the point(s) of inflection.

$f''(x) = -6e^{-3x^2} + 36x^2e^{-3x^2} = 6e^{-3x^2}(-1 + 6x^2)$

$f''(x) = 0$ for $6x^2 = 1 \Rightarrow x^2 = 1/6 \Rightarrow x = \pm\sqrt{1/6} = \pm 1/\sqrt{6}$

$f''(x)$: + 0 − 0 +

x: −1 Test, $\boxed{-1/\sqrt{6}}$, 0 Test, $\boxed{1/\sqrt{6}}$, 1 Test

Points of inflect. at $x = \pm\frac{1}{\sqrt{6}}$

$y = e^{-3(\frac{1}{\sqrt{6}})^2} = e^{-\frac{1}{2}}$

Interval(s) on which f is concave up: $(-\infty, -1/\sqrt{6})$ and $(1/\sqrt{6}, \infty)$

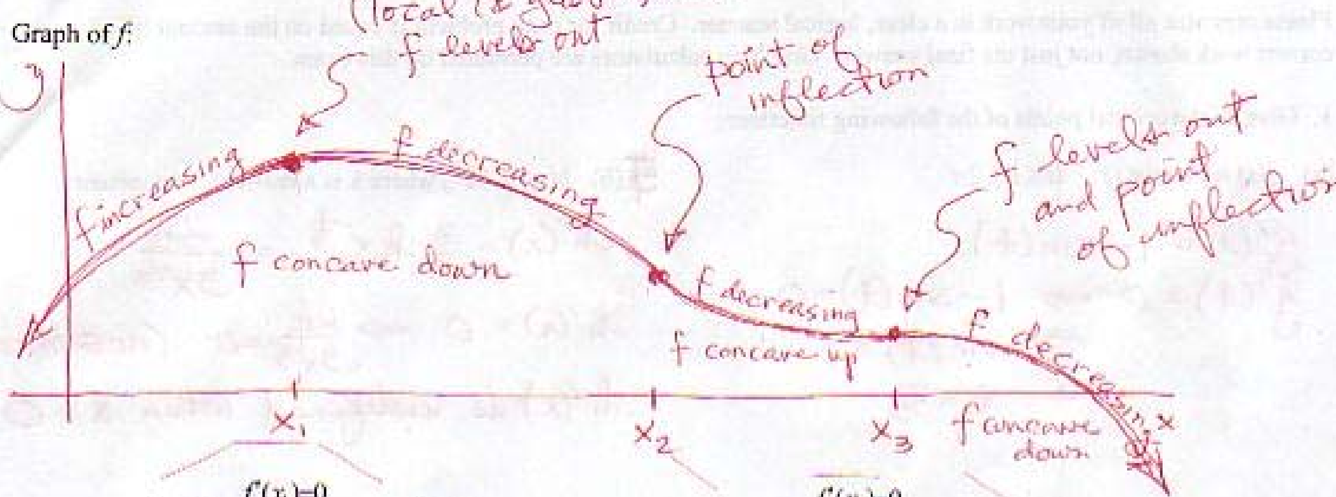
Interval(s) on which f is concave down: $(-1/\sqrt{6}, 1/\sqrt{6})$

Point(s) of Inflection (Give both the x and y value): $(-1/\sqrt{6}, e^{-1/2})$ $(1/\sqrt{6}, e^{-1/2})$

No points off for including [illegible] in part (b)

9 3. Given the following sign charts for the first and second derivatives of a function, f, sketch out a possible graph of f.

Graph of f:



$f'(x_1)=0$ $\qquad$ $f'(x_3)=0$

$f'(x)>0$ $\quad x_1 \quad$ $f'(x)<0$ $\quad x_2 \quad$ $f'(x)<0$ $\quad x_3 \quad$ $f'(x)<0$

$f''(x_2)=0$ $\qquad$ $f''(x_3)=0$

$f''(x)<0$ $\quad x_1 \quad$ $f''(x)<0$ $\quad x_2 \quad$ $f''(x)>0$ $\quad x_3 \quad$ $f''(x)<0$

7 4. For some positive constant, C, a patient's temperature change, T, due to a dose, D, of a drug is given by

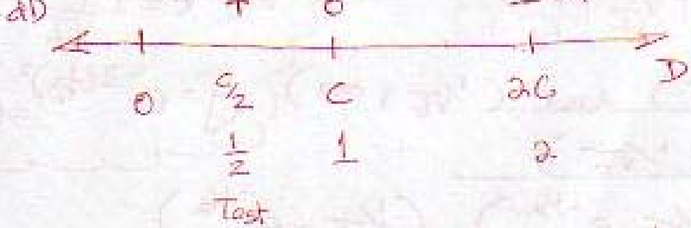
$$T = \frac{CD^2}{2} - \frac{D^3}{3}$$

Realistically, $D \geq 0$ since you can't have a negative amount, or dose, of medicine

Use calculus to determine what dose, D, will maximize temperature change, T.

$\frac{dT}{dD} = CD - D^2 = 0$ for $CD - D^2 = 0$
$= D(C-D)$ $\qquad D(C-D)=0$

$D=0 \quad D=C \Leftarrow$ CRITICAL POINTS



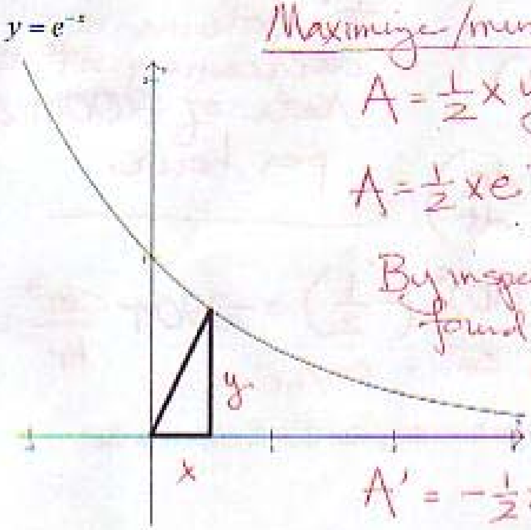
(take $C=1$ to establish a pattern)

The dose that maximizes temperature is $\underline{C}$, Verified by T' sign chart.

positive

5. A right triangle has one vertex at the origin, one vertex on the curve $y = e^{-x}$, its base on the x-axis and one side parallel to the y-axis (see picture). Use calculus to find the maximum and minimum areas for such a triangle. Give exact values for your answers.

8



Maximize/minimize area

$A = \frac{1}{2} x y$

$A = \frac{1}{2} x e^{-x}, \; x \geq 0$

By inspection, min A is found when $x = 0$ (the area would be zero)

Constraint

$y = e^{-x}$

Report: The max area is $\frac{1}{2}e^{-1}$ unit^2

$A' = -\frac{1}{2}xe^{-x} + \frac{1}{2}e^{-x} = 0$

$e^{-x}\left(-\frac{1}{2}x + \frac{1}{2}\right) = 0$

c.p. $x = 1$

for $x < 1$ A' is +

for $x > 1$ A' is −

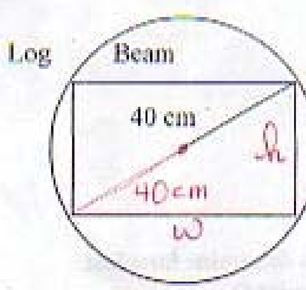
∴ $x = 1$ locates a global max

8

6. A rectangular beam is cut from a cylindrical log of radius 40 cm. The strength of a beam of width w and height h is proportional to wh^2. Use calculus to find the width and height of the beam of maximum strength.

Report: The width should be $\frac{80}{\sqrt{3}}$ cm and the length (height) should be $\sqrt{\frac{12800}{3}}$ cm

Cross-sectional view of log and beam:



S = strength (units?)
w = width (cm)
h = height (cm)

$S = k w h^2$ $\quad$ k = constant of proportionality

$S = kw(6400 - w^2)$

$S = k6400w - kw^3$

DOMAIN: $0 \leq w \leq 80$

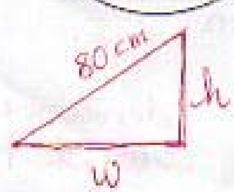
$S' = 6400k - 3kw^2 = 0$

$6400k - 3kw^2 = 0$

$w^2 = \frac{6400}{3}$

$w = \pm\sqrt{\frac{6400}{3}} = \frac{80}{\sqrt{3}}$ (negative value is out of domain)

w	S
0	0
$80/\sqrt{3}$	$k\frac{80}{\sqrt{3}}\left(6400 - \frac{6400}{3}\right)$ MAX!
80	0



$w^2 + h^2 = 80^2$

$w^2 + h^2 = 6400$

$h^2 = 6400 - w^2$

$h^2 = 6400 - \left(\frac{80}{\sqrt{3}}\right)^2$

$= 6400 - \frac{6400}{3} = \frac{12800}{3} \Rightarrow h = \sqrt{\frac{12800}{3}}$

7 7. A spherical snowball is melting. Its radius is decreasing at a rate of ½ cm per hour when the radius is 10 cm. Use calculus to determine how fast the volume is decreasing at that time.

$V=\frac{4}{3}\pi r^3$

$\frac{dr}{dt}=-\frac{1}{2}\frac{cm}{hr}$ when $r=10$ cm

$$\left\{\begin{aligned} V&=\frac{4}{3}\pi r^3\\ \frac{d}{dt}(V)&=\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)\\ \frac{dV}{dt}&=\frac{4}{\cancel{3}}\pi\left(\cancel{3}r^2\frac{dr}{dt}\right)\end{aligned}\right.$$

$$\frac{dV}{dt}=4\pi(10)^2\left(-\frac{1}{2}\right)=-200\pi\ \frac{cm^3}{hr}$$

$cm^2 \cdot cm/hr$

Report: The volume is decreasing at a rate of 200π cm^3 per hour.

8. A lighthouse is 3 kilometers from a long, straight coastline.

6 (a) Use calculus to find the rate of change of the distance of the spot of light from the point O with respect to the angle θ.

x

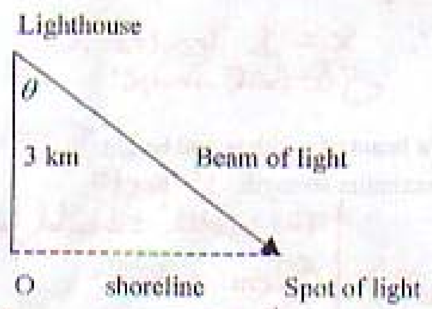
Find $\frac{dx}{d\theta}$:

$\frac{x}{3}=\tan\theta$

$x=3\tan\theta$

$$\boxed{\frac{dx}{d\theta}=3\sec^2\theta}$$

OR $$\boxed{\frac{dx}{d\theta}=\frac{3}{\cos^2\theta}}$$



x = distance (km) from O to spot of light.

4 (b) If the light in the lighthouse is rotating at a rate of 1 revolution per minute, use calculus to determine how fast the spot of light is moving along the shore at the moment when it (the spot) is 3 km from the point O.

x x

derivative w.r.t. time

$\frac{1\text{ revolution}}{\text{min}}=\frac{2\pi\text{ radians}}{\text{min}}=\frac{d\theta}{dt}$

Find $\frac{dx}{dt}$ when $x=3$ km, given $\frac{d\theta}{dt}=2\pi\ \frac{\text{rads}}{\text{min}}$

By the chain rule, we have

$$\frac{dx}{dt}=\frac{dx}{d\theta}\cdot\frac{d\theta}{dt}=3\sec^2\theta\cdot\frac{2\pi\text{ radians}}{\text{min}}$$

$$=3(2)^{(*)}\cdot 2\pi\frac{\text{rads}}{\text{min}}=\boxed{12\pi\frac{\text{rads}}{\text{min}}}$$

(*) but $x=3$ km means $\tan\theta=1$ means $\theta=\frac{\pi}{4}$ means $\sec\theta=\sqrt{2}$, $\sec^2\theta=2$

Math 265A: Test 4, Part II **Name:** KEY

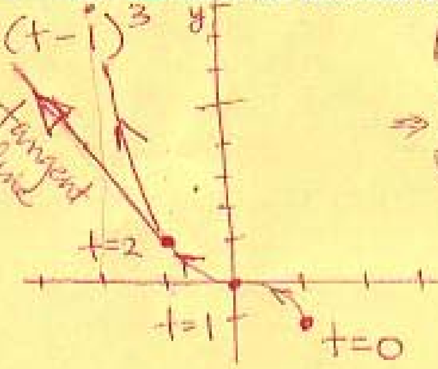
Open Note Portion: You may use your notes for help in solving the following problems.

The position of a particle moving on a path is given parametrically as $x = 1 - t$, $y = (t-1)^3$ for $0 \le t \le 3$ where t = time in seconds and x, y are in centimeters.

(a) (4 pts) Sketch a graph of the path of the particle. Include time labels and directional arrows.

t	x	y
0	1	-1
1	0	0
2	-1	1
3	-2	8

$x = 1 - t,\ y = (t-1)^3$



Note: since $t = 1 - x$
$\Rightarrow y = (1 - x - 1)^3$
$\Rightarrow y = -x^3$
so the graph is a cubic curve.

(b) (6 pts) Determine the velocity of the particle in the x-direction and y-direction. Explain what each velocity means in terms of how the particle is moving.

x-direction: velocity $= \frac{dx}{dt} = -1$ cm/sec. The particle is moving left at a constant rate.

y-direction: velocity $= \frac{dy}{dt} = 3(1-t)^2 \frac{cm}{sec}$ The particle is moving up at an increasing rate, after $t = 1$. Before $t = 1$ the particle is slowing down. At $t = 1$ it is not moving in the y-direction at all.

(c) (3 pts) Does the particle ever completely stop moving? How can you tell?

No. Because $\frac{dx}{dt} \neq 0$ for any t value, so will never have both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ equal to zero at the same time.

(d) (7 pts) Find the equation of the tangent line (as an xy-equation) at time t = 2. Sketch the tangent line on the graph in part (a) and explain what this line means in terms of movement of the particle.

$$m_{tan}\Big|_{t=2} = \frac{dy/dt}{dx/dt}\Big|_{t=2} = \frac{3(1-t)^2}{-1}\Big|_{t=2} = -3$$

Equation: $y - 1 = -3(x+1)$
$y = -3x - 2$

The tangent line gives the path the particle would follow if the forces that are keeping it on its path were lifted (~~the~~ removed).