Math 265B Spring 2020 **Name:** KEY
100 points

INSTRUCTIONS: For full or partial credit, all solutions must be neatly written with all of the details clearly presented. Answers given without adequate justification may not receive credit. Make sure you are using correct notation. Box answers where appropriate.

1. (2 pts) What is the difference between a <u>sequence</u> and a <u>series</u>?

A sequence is just a list of numbers (<u>ordered</u>)

A series is a list of numbers being added together.

2. (4 pts) (a) What does it mean for a sequence to converge?

A sequence $\{a_n\}_{n=1}^{\infty}$ converges if $\lim\limits_{n\to\infty} a_n$ exists.

(b) What does it mean for a series to converge?

A series $\sum\limits_{k=1}^{\infty} a_k$ converges if the sequence of partial sums, $S_n = \sum\limits_{k=1}^{n} a_k$ Converges.

3. (10 pts) Consider the series $\sum\limits_{k=1}^{\infty} \frac{1}{2k+1} - \frac{1}{2k+3}$

3 (a) Find the first 3 partial sums for the series

$$S_1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

$$S_2 = \underline{\frac{1}{3} \cancel{- \frac{1}{5}}} + \underline{\cancel{\frac{1}{5}} - \frac{1}{7}} = \frac{1}{3} - \frac{1}{7} = \frac{4}{21}$$

$$S_3 = \underbrace{\frac{1}{3} - \frac{1}{5}}_{k=1} + \underbrace{\frac{1}{5} - \frac{1}{7}}_{k=2} + \underbrace{\frac{1}{7} - \frac{1}{9}}_{k=3} = \frac{1}{3} - \frac{1}{9} = \frac{6}{27} = \frac{2}{9}$$

1 (b) What is the name of this type of series? Telescoping Series

3 (c) Find a formula for the nth partial sum of the series

$$S_n = \frac{1}{3} \cancel{- \frac{1}{5}} \cancel{+ \frac{1}{5}} \cancel{- \frac{1}{7}} \cancel{+ \frac{1}{7}} \cancel{- \frac{1}{9}} \cancel{+} \cdots \cancel{+ \frac{1}{2n+1}} - \frac{1}{2n+3}$$

$$\boxed{S_n = \frac{1}{3} - \frac{1}{2n+3}}$$

3 (d) Does the series converge or diverge? If it converges, what value does it converge to? (Your work should include a limit!)

The series converges to $\frac{1}{3}$; i.e $\boxed{\sum\limits_{k=1}^{\infty} \frac{1}{2k+1} - \frac{1}{2k+3} = \frac{1}{3}}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{2n+3}\right) = \frac{1}{3}$$

4. (5 pts) True or false (circle the correct answer):

pos Smaller bigger

(a) If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} b_n$ converges also. True [✓ False]

(b) If $a_n \le b_n \le 0$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges also. [✓ True] False — this one is tricky — look at a graph (see (*))

neg!

(c) For a series $\sum_{n=1}^{\infty} a_n$ if you find that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, then you can conclude the series diverges. True [✓ False]

← inconclusive result from Ratio Test

(d) The interval of convergence of a power series gives the x-values for which the series ~~diverges~~: True [✓ False]

Converges

(e) The formula for Taylor polynomials was created by making a function and its derivatives and the polynomial and its derivatives match at a particular x-value. [✓ True] False

5. (15 pts) Fill in and/ or circle the correct answer:

2 (a) $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ [✓ converges] / diverges (circle one) because p-Series, $p = \frac{3}{2} > 1$

2 (b) $\sum_{k=0}^{\infty} \frac{1}{3}\left(\frac{5}{4}\right)^k$ converges / [✓ diverges] (circle one) because Geometric Series $|r| = \frac{5}{4} > 1$

2 (c) If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then $\sum_{k=1}^{\infty} |a_k|$ must converge / [✓ diverge] (circle one)

2 (d) The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the Harmonic series and it converges / [✓ diverges] (circle one).

3 (e) Does the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converge absolutely? No Show work or explain how you know (no proof necessary!).

$\sum |a_K| = \sum \left|\frac{(-1)^K}{K}\right| = \sum \frac{1}{K}$ — which diverges

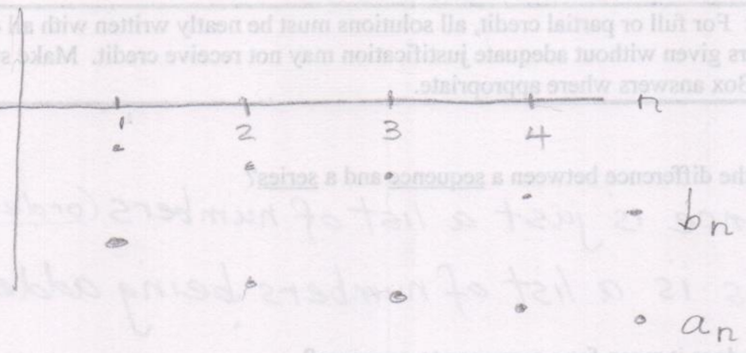
3 Does $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converge conditionally? Yes Show work to justify your answer.

Alternating Series Test: Terms are decreasing, because $\frac{1}{K} < \frac{1}{K+1}$

and $\lim_{K\to\infty} \frac{1}{K} = 0$ ∴ Series converges.

(*)

#4(b)



$a_n \le b_n \le 0$

means

a_n and b_n are both negative

AND the b_n values are closer to zero, so smaller in magnitude than a_n values

so if $\sum b_n$ diverges, it must go to $-\infty$

and since $\sum a_n$ is "larger" (in magnitude) than $\sum b_n$

it must be that $\sum a_n$ diverges also

6. (6 pts) Find the sum of the series $\sum_{k=0}^{\infty} \frac{2^k+7}{3^k}$ $= \sum_{K=0}^{\infty} \frac{2^K}{3^K} + 7\left(\frac{1}{3^K}\right)$

$$= \sum_{K=0}^{\infty} \left(\frac{2}{3}\right)^K + 7\left(\frac{1}{3}\right)^K$$

Since both Series converge, we can write $\sum a_K + b_K$ as $\sum a_K + \sum b_K$ $\longrightarrow$ $= \sum_{K=0}^{\infty} \left(\frac{2}{3}\right)^K + \sum_{K=0}^{\infty} 7\left(\frac{1}{3}\right)^K$

$$= \frac{1}{1-\frac{2}{3}} + \frac{7}{1-\frac{1}{3}}$$

$$= 3 + \frac{21}{2} = \boxed{\frac{27}{2}} \text{ or } \boxed{13.5}$$

7. (6 pts) Use the Integral Test to <u>prove</u> that the series either converges or diverges. Be sure to check the conditions (show work!) and clearly state your conclusion.

$$\sum_{k=1}^{\infty} \frac{2k}{e^{k^2}}$$

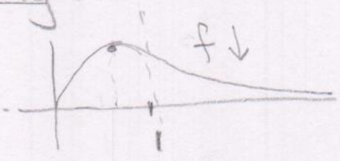
Conditions: $f(x) = \frac{2x}{e^{x^2}}$

1. f is continuous? - yes (no division by zero!

2. f is positive for $x \geq 1$? $f(x) \geq 0$ yes, since $2x > 0$ for $x \geq 1$ and $e^{x^2} \geq 0$ for all x.

3. f is decreasing for $x \geq 1$?

By graph:



By calc 1:

$$f'(x) = \frac{2e^{x^2} - 2x(2xe^{x^2})}{(e^{x^2})^2}$$

$$= \frac{2e^{x^2}(1-2x^2)}{(e^{x^2})^2} = 0$$

$1 - 2x^2 = 0 \quad x = \pm\frac{1}{\sqrt{2}}$

$$f' \quad + \quad \Big| \quad - \qquad \frac{1}{\sqrt{2}}$$

So f is decreasing on $[\frac{1}{\sqrt{2}}, \infty)$

$$\int_1^{\infty} \frac{2x}{e^{x^2}}\,dx$$

$$= \lim_{b \to \infty} \int_1^b 2xe^{-x^2}\,dx$$

$$= \lim_{b \to \infty} \left[-e^{-x^2} \Big|_1^b \right]$$

$$= \lim_{b \to \infty} \left[-e^{-b^2} + e^{-1} \right]$$

$= 0 + \frac{1}{e} = \frac{1}{e} \Rightarrow$ integral converges

$\therefore$ the series converges

a_K

8. (6 pts) Use the Limit Comparison Test to prove that the series $\sum_{k=1}^{\infty}\frac{3k^2+k}{k^5+5k^2+2}$ converges or diverges.

Clearly state your conclusion!

Compare to $\sum_{K=1}^{\infty}\frac{1}{K^3}$ b_K

Conditions: $a_K, b_K > 0$?
Yes for $K \geq 1$

Note:
$\sum \frac{1}{K^3}$ converges
by the p-Test
$p = 3 > 1$

Test: $\lim_{K\to\infty}\frac{a_K}{b_K}$

$$= \lim_{K\to\infty}\frac{\frac{3k^2+k}{K^5+5K^2+2}}{\frac{1}{K^3}}$$

$$= \lim_{K\to\infty}\frac{3k^5+K^4}{K^5+5K^2+2} = 3$$

$\Longrightarrow$ Since the limit exists and $\sum\frac{1}{K^3}$ converges
the original series $\boxed{\text{converges}}$

9. (4 pts) Use any test to prove the series $\sum_{k=1}^{\infty}\frac{5k^3}{k^3+10k}$ converges or diverges. Clearly state your conclusion.

Divergence Test:
Conditions: none! :)

$$\lim_{K\to\infty}\frac{5k^3}{K^3+10K} = 5 \neq 0$$

Since the terms are not decreasing to zero
the series $\boxed{\text{diverges}}$

Note: As K gets large, $a_K \approx 5$ so the series
becomes basically $\sum_{K=\text{large}}^{\infty} 5 = \underline{5+5+5+5+\cdots}$

This is definitely
headed off to
infinity!

10. (8 pts) (a) Write the following series using $\sum$ form: $\frac{1}{2}-\frac{3}{4}+\frac{5}{8}-\frac{7}{16}+\cdots$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k-1)}{2^k}$$ (answers will vary — be sure to check a few terms by expanding!)

check: $$\underset{k=1}{\frac{(-1)^2(2(1)-1)}{2^1}} + \underset{k=2}{\frac{(-1)^3(2(2)-1)}{2^2}} + \cdots = \frac{1}{2} - \frac{3}{4} + \cdots$$

checks out!

(b) Determine whether the series converges or diverges, then prove your guess using an appropriate test.

By inspection, we see this is an alternating series with terms going to zero, so it will at least converge conditionally by Alternating Series Test. To check for absolute convergence, let's use Ratio Test.

Ratio Test:

Note:
Using Alternating Series Test is okay since I didn't ask about Absolute Convergence.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{\frac{2(k+1)-1}{2^{k+1}}}{\frac{2k-1}{2^k}}\right|$$

$$= \lim_{k\to\infty} \frac{2k+1}{\cancel{2^k}\cdot 2^1}\cdot\frac{\cancel{2^k}}{2k-1}$$

$$= \lim_{k\to\infty} \frac{2k+1}{2(2k-1)} = \frac{1}{4} < 1 \Rightarrow$$ By the Ratio Test the series <u>converges</u> absolutely.

11. (12 pts) Given the following power series, determine the radius and interval of convergence (you do not have to test the endpoints of the interval).

(a) $\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(2(n+1))!}}{\frac{x^n}{(2n)!}} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{x^n \cdot x^1}{(2n+2)!} \cdot \frac{(2n)!}{x^n} \right| < 1$$

$$|x| \cdot \lim_{n\to\infty} \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \Bigg| < 1$$

This will be true for all x-values! ⟶

$$|x| \cdot 0 < 1$$

R = ∞ Interval = $(-\infty, \infty)$

(b). $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{n+1}}{\frac{(x-2)}{n}} \right| < 1$$

$$\lim_{n\to\infty} |x-2| \cdot \frac{n}{n+1} < 1$$

$$|x-2| \cdot \lim_{n\to\infty} \frac{n}{n+1} < 1$$

$$|x-2| \cdot 1 < 1$$

$$|x-2| < 1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

Note: If you tested end points you'd get convergence for $x=1$ and divergence for $x=3$

R = 1 Interval = $(1, 3)$

$[1, 3)$

12. (6 pts) Use the Taylor Polynomial (derivative) formula to find the Taylor polynomial of degree 3 about x = 2 for the function $\ln(x)$

$f(x) = \ln(x)$ | $f(2) = \ln 2$

$f'(x) = \frac{1}{x} = x^{-1}$ | $f'(2) = \frac{1}{2}$

$f''(x) = -x^{-2} = -\frac{1}{x^2}$ | $f''(2) = -\frac{1}{4}$

$f'''(x) = 2x^{-3} = \frac{2}{x^3}$ | $f'''(2) = \frac{2}{8} = \frac{1}{4}$

$$f(x) \approx P_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)(x-2)^2}{2!} + \frac{f'''(2)(x-2)^3}{3!}$$

$$\boxed{= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3}$$

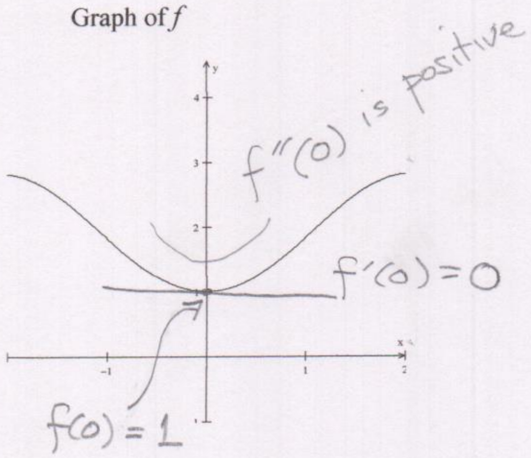
13. (3 pts) Suppose $P_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function f about x = 0. What can you say about value of a, b and the sign of c, based the graph of f?

Value of a: 1 (f(0) = 1)

Value of b: 0 (f'(0) = 0)

Sign of c: positive (f''(0) > 0)

Graph of f



14. (3 pts) Write down (you do not have to find them from the derivative formula!) the Taylor series (aka MacLaurin series), expanded about x = 0, for the following functions. Write them in expanded form and using Σ notation.

a. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

b. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

c. $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

15. (10 pts) (a) Use the Taylor series for e^x to determine the Taylor Series for $e^{i\theta}$. Find the first 7 terms of the series and simplify the result. You don't need to write the general term or use sigma notation.

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

Powers on i:

$i^2 = -1$ $\quad i^4 = (-1)(-1) = 1$ $\quad i^6 = i^2 \cdot i^4 = -1$

$i^3 = i^2 \cdot i = -i$ $\quad i^5 = i^4 \cdot i = i$ $\quad$ etc

(b) Partition the series above into real terms and imaginary* terms (*the terms with an i in them after simplifying)

$$e^{i\theta} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots}_{\text{Real}} \underbrace{+ i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \cdots}_{\text{Imaginary}}$$

(c) Identify the series for the real part and for the imaginary part (factor out the i in the imaginary terms first).

$$1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \cos(\theta)$$

$$i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = i\sin\theta$$

(d) Based on the work above, write an equation that relates $e^{i\theta}$, $\sin\theta$, and $\cos\theta$

$e^{i\theta} = \cos\theta + i\sin\theta$

(This relationship is called "Euler's Equation".