Math 265B: Chapter 9 [Chapter 10] Convergence Test Theorems

Theorem (The Divergence Test):

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum a_k$ diverges. *(This is the case where the terms aren't getting small as the series grows.)*

Theorem (The Geometric Series Test):

The geometric series $\sum_{k=0}^{\infty} a \cdot r^k$ converges to $\dfrac{a}{1-r}$ if and only if $|r| < 1$,

Theorem (The Integral Test):

If $f(x)$ is a *continuous*, *positive*, *decreasing* function, defined for $x \geq c$, where c is some real number, and if $f(\text{k}) = a_k$ for all $k \geq c$, then the improper integral $\int_c^{\infty} f(x)dx$ and the series $\sum_{k=1}^{\infty} a_k$ either both converge or both diverge. Note: To use this test, you must check that the conditions on f are satisfied; i.e., verify that $f(x)$ is continuous, positive, and decreasing.

Theorem (The *p*-Series Test):

The series $\sum_{k=1}^{\infty} \dfrac{1}{k^p}$ is convergent for $p > 1$, and divergent for $p \leq 1$.

Theorem (Direct Comparison Test): (Section 9.5) Suppose $0 \leq a_k \leq b_k$ for all $n \geq N$.

(i) If $\sum_{k=1}^{\infty} b_k$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is also convergent.

(ii) If $\sum_{k=1}^{\infty} a_k$ is divergent, then $\sum_{k=1}^{\infty} b_k$ is also divergent.

Theorem (Limit Comparison Test): Suppose $a_k, b_k > 0$ for all k $\geq N$. If $\lim_{k\to\infty} \dfrac{a_k}{b_k} = L$ for some finite, positive number L, then either both series $\sum a_k$, $\sum b_k$ converge or both diverge.

Theorem (The Ratio Test): (Section 9.5) Suppose $a_k \neq 0$ for all k, and $\lim_{k\to\infty} \left| \dfrac{a_{k+1}}{a_k} \right| = L$.

If $L < 1$, then the series $\sum a_k$ converges.

If $L > 1$, then $\sum a_k$ diverges (L may be ∞).

If $L = 1$, the test is inconclusive.

Theorem (Alternating Series Test): (Section 9.6) If $0 < a_{k+1} \leq a_k$ for all k, and $\lim_{k\to\infty} a_k = 0$, then the alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$ converges.