Note: We have solved and graphed slope fields for DE's of the forms below previously. Try putting in numbers for the constants to visualize the slope fields.

Math 265B: Differential Equations Applications

For each of the descriptions below, set up the DE, sketch a slope field with examples of solution curves, and find any equilibrium solutions. Classify the equilibrium solutions as stable or unstable. Then use Wolfram to solve the DE.

Population Growth Models

Exponential Growth: Assume there are no limitations of resources which leads to unbounded growth…proposed by the economist Thomas Malthus). *(Interesting discussion on the topic in the era of COVID-19: https://www.intelligenteconomist.com/malthusian-theory/)*

The rate at which a population grows is directly proportional to the actual population size at any time, t.

P = population at time, t

$\frac{dP}{dt}$ = rate of growth

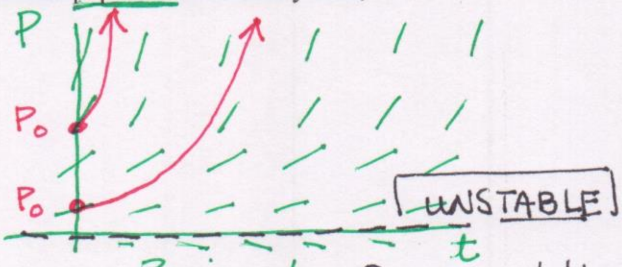
t = time (units could be hrs for bacteria, months for bunnies, years for humans, etc.)

$$\frac{dP}{dt} = rP$$

$$\boxed{P = C_0 e^{rt}}$$

C_0 = initial pop $= P_0$

Equil. $\frac{dP}{dt} = 0$ for $P = 0$



neg P is not meaningful

$P=0$, unstable equilibrium

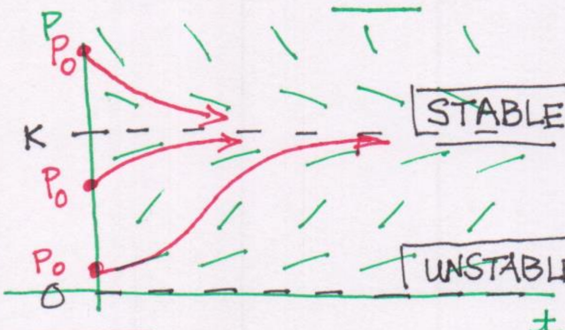
*Note: The constant of proportionality is called the "growth factor' in this context and is usually denoted by "r". It isn't the growth **rate**, per se, because the actual rate depends on the population size though we often call r a growth rate. Just FYI.*

Logistic Growth: When there are limitations on resources, the growth rate is attenuated (dialed down) by a factor that takes into account the "carrying capacity", which is the largest sustainable population size.

The rate at which a population grows is jointly proportional to the size of the population and the factor $\left(1-\frac{P}{K}\right)$, where K = the carrying capacity.

$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right)$$

Equilibrium: $\frac{dP}{dt} = 0$ when $\boxed{P=0}$ and $1-\frac{P}{K}=0$ so $\boxed{P=K}$



When $P \ll K$ then $1-\frac{P}{K} \approx 1$ so you get exponential growth

As P increases and approaches the carrying capacity, K, then $\frac{P}{K} \to 1$ so $\left(1-\frac{P}{K}\right) \to 0$, thus slowing down growth.

$$\cancel{P = \frac{c^{rt}}{c_1 + e^{rt}}}$$

Bah! Wrong answer!

Wolfram gives an odd form for solution

correct

$$\boxed{P = \frac{K}{1+Ae^{-rt}}}$$

Epidemiology: When there is an infectious disease circulating in the population, the rate of infection is based on the interaction of how many people are infected and how many people are vulnerable but not yet infected. This turns out to also be a logistic model.

The rate of infection is proportional to the product of the number of people infected and the number of vulnerable people not yet infected.

I = number infected

N = total who are vulnerable

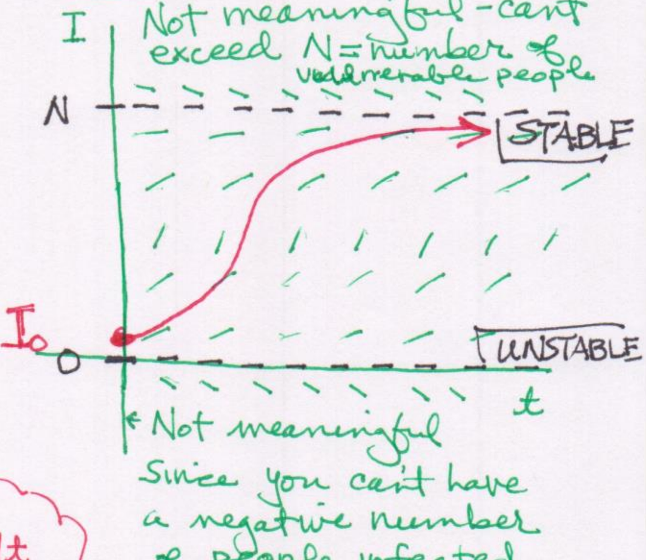
$N-I$ = number who are still not yet infected

$$\frac{dI}{dt} = kI\cdot(N-I)$$

Equil. solutions

$\frac{dI}{dt} = 0$ when

$I=0$ or $I=N$



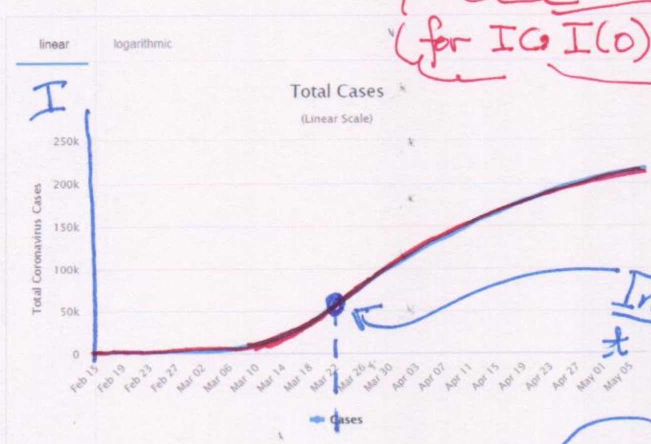
Note: This DE would be solved using Partial Fractions - we did that once which is enough...

Wolfram isn't helpful here without hacking :(

$$I = \frac{I_0 N}{I_0 + (N-I_0)e^{-kNt}}$$

(for IC $I(0) = I_0$)

Total Coronavirus Cases in Italy

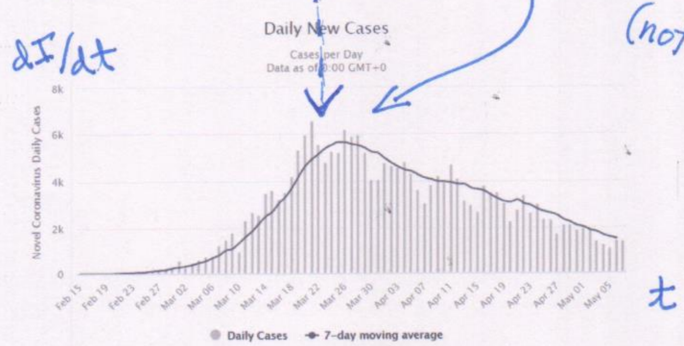


Actual COVID-19 case curve from Italy

Inflection Point

Corresponds to the peak of the rate curve (not the slope field)

Daily New Cases in Italy



Decay Models. Both **radioactive decay** and **elimination of a drug** from a person's body are modeled the same way. *(Note: Half-life is a key characteristic of both radioactive decay and drug elimination. Dosing of drugs is very dependent on half-life, with more caution taken with drugs with a longer half-life since they remain in the system longer and can build up to toxic levels with repeated dosing.)*

The rate of decay of a radioactive element is proportional to the amount present at time t.

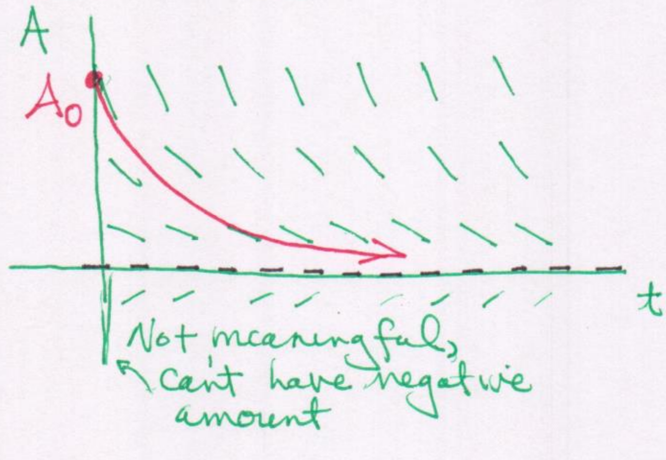
A = amount at time t

$$\frac{dA}{dt} = -kA$$

Equil. Solution

$A = 0$

$\frac{dA}{dt} < 0$ for any $A > 0$



$$A = A_0 e^{-kt} \quad \text{given } A(0) = A_0$$

Newton's Law of Heating and Cooling:
Newton proposed that the temperature of a hot object changes at a rate proportional to the difference between its temperature and that of its surroundings. Similarily, a cold object heats up at a rate proportional to the temperature difference between the object and the ambient (surrounding) temperature.

H = temperature of object

A = ambient temperature

$$\frac{dH}{dt} = k(H - A)$$

Equil: $\frac{dH}{dt} = 0$ when $H = A$

For I.C. of $H(0) = H_0$

$$\boxed{H = A + (H_0 - A)e^{-kt}}$$