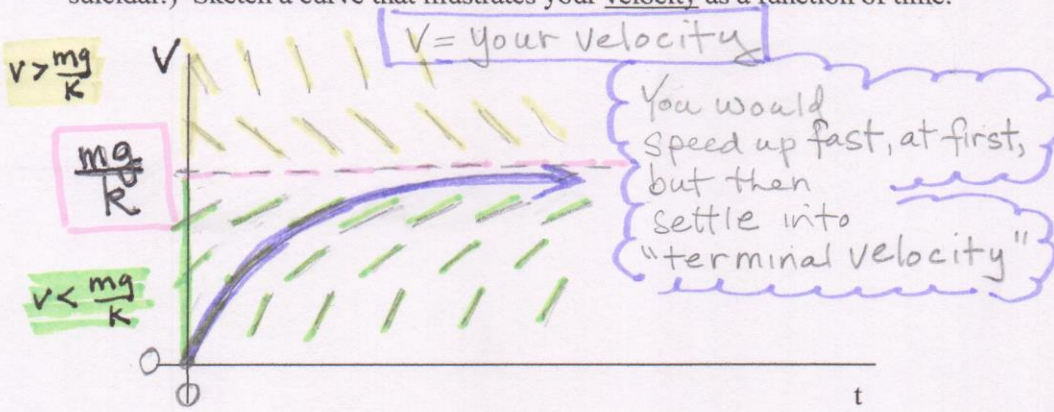
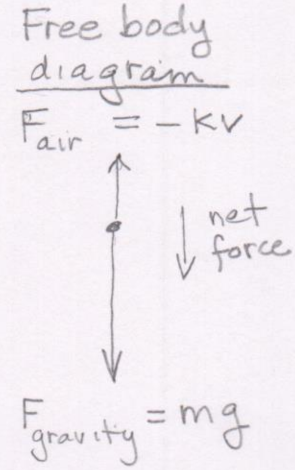
Math 265B: Free Fall with Air Resistance

This exploration was shared with me by a physics (not math) professor at another college.

Suppose you've just jumped off of a very high cliff. (Imagine you're a base-jumper with a wingsuit…not suicidal!) Sketch a curve that illustrates your <u>velocity</u> as a function of time.





Note: Choose the positive direction to be downward

Now, the air is resisting you as you fall. What type of physical quantity is air resistance? I.e., is it a mass? A distance? A force?

What is air resistance proportional to? Your velocity

So $F_{air} =$ kv, where k = the constant of proportionality

(As an aside, think about what physical factors k depends on; i.e., what would increase or decrease air resistance?)

We can say that the two forces at play, then, are the force of gravity

and the force of the air resistance.

Set up an equation to model the relationship and net effect of these forces:

ma	=	mg	+	$-kv$
Net Force (think Newton's Second Law)		*Force of gravity*		*Force of the air resistance*

But what is a in terms of v? $a = dv/dt$ Substitute this into the equation, and voila! We have a First Order DE, and with the addition of an Initial Condition, we'll have an IVP.

$m\frac{dv}{dt} = mg - kv$ (Differential Equation) + $v(0) = 0$ Initial Condition

Recopy the DE and solve for dv/dt. Then find the equilibrium solution.

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

Equil. Solution

$$\frac{dv}{dt} = 0$$

$$g - \frac{k}{m}v = 0$$

$$\Rightarrow \boxed{v = \frac{mg}{k}}$$

Slope field

$\frac{dv}{dt} > 0$ /

$g - \frac{k}{m}v > 0$

$v < \boxed{\frac{mg}{k}}$

$\frac{dv}{dt} < 0$ \

$v > \boxed{\frac{mg}{k}}$

Show the equilibrium solution on the graph.

Now fill in the slope field above and below the equilibrium solution. Your original concept sketch is now the solution curve to the I.V.P.

Solve the D.E. using Separation of Variables. Then solve the I.V.P. (evaluate C using the I.C.)

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

$$\frac{dv}{dt} = -\frac{k}{m}\left(v - \frac{mg}{k}\right)$$

$$\int \frac{dv}{v - \frac{mg}{k}} = \int -\frac{k}{m}\,dt$$

$$\ln\left|v - \frac{mg}{k}\right| = -\frac{k}{m}t + C$$

$$e^{\ln\left|v - \frac{mg}{k}\right|} = e^{-\frac{k}{m}t + C}$$

$$\left|v - \frac{mg}{k}\right| = e^{C}\cdot e^{-\frac{k}{m}t}$$

$$v - \frac{mg}{k} = \underbrace{\pm e^{C}}\, e^{-\frac{k}{m}t}$$

$$v = \frac{mg}{k} + \underbrace{C_1}\, e^{-\frac{k}{m}t}$$

IC: $v(0) = 0$

$$\Rightarrow 0 = \frac{mg}{k} + C_1 \Rightarrow C_1 = -\frac{mg}{k}$$

Follow up questions:

(1) What is the steady state for the velocity function?

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$$

$$= \frac{mg}{k}(1 - 0) = \boxed{\frac{mg}{k}}$$

$$v(t) = \frac{mg}{k} - \frac{mg}{k}e^{-\frac{k}{m}t}$$

$$v(t) = \boxed{\frac{mg}{k}}\left(1 - e^{-\frac{k}{m}t}\right)$$

(2) **Extension.** In Intro Physics classes, they neglect air-resistance when studying free fall and projectile motion. Free-fall in a vacuum gives this simple velocity formula: $v = gt$

Use the MacLaurin Series for e^x to convert the velocity formula we found into a series, then explain why, for small values of t, you get $v \approx gt$. What does this mean physically?

(2) Show $v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \approx gt$ for small t.

Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

so $e^{-\frac{k}{m}t} = 1 + \left(-\frac{k}{m}t\right) + \frac{\left(-\frac{k}{m}t\right)^2}{2!} + \frac{\left(-\frac{k}{m}t\right)^3}{3!} + \cdots$

so $1 - e^{-\frac{k}{m}t} = 1 - \left[1 - \frac{k}{m}t + \frac{\left(\frac{k}{m}t\right)^2}{2} - \frac{\left(\frac{k}{m}t\right)^3}{6} + \cdots\right]$

$= \frac{k}{m}t - \frac{\left(\frac{k}{m}t\right)^2}{2} + \frac{\left(\frac{k}{m}t\right)^3}{6} - \cdots$

Okay this is how physicists reason:
if t is small, say $t = .1$
then $t^2 = .01$ is significantly smaller
and $t^3 = .001$ is even smaller, so
physicists say, "Bah, those terms are
negligible, so let's just drop them!"

So, for small t, $1 - e^{-\frac{k}{m}t} \approx \frac{k}{m}t$ (drop all the other terms with $t^2, t^3, \ldots$)

So, $v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \approx \frac{mg}{k}\left(\frac{k}{m}t\right) = gt$.

$v \approx gt$ for small t.

For the first moments, your velocity isn't enough for air resistance to be much of a factor, so it's like free-fall in a vacuum.