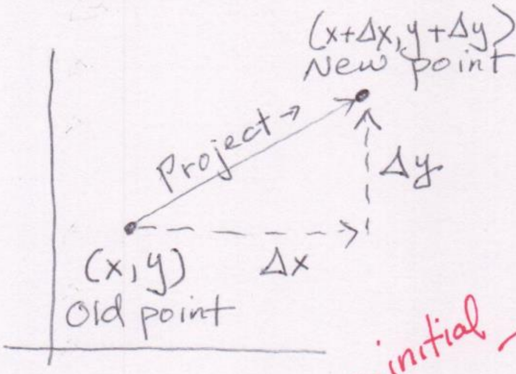
Math 265B: Euler's Method

Initial Condition

Final x (x=2) What is y, approximately, at x=2

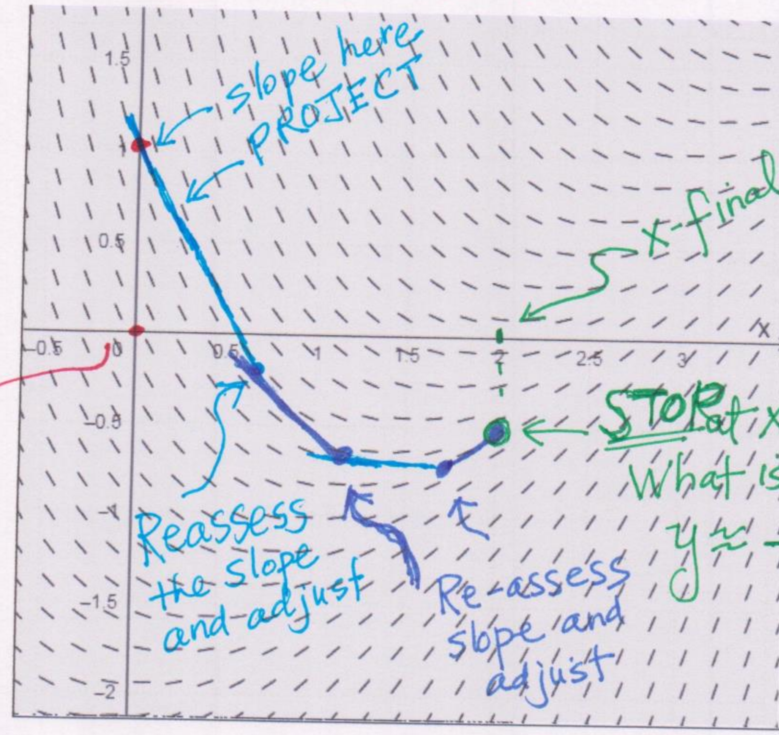
Goal: Estimate the solution to the IVP $y'(x) = x - y - 2$, $\boxed{y(0) = 1}$ at y(2)

Make a "Polygonal Approximation" of the solution curve



$$m = \frac{\Delta y}{\Delta x}$$

So $\Delta y = m \cdot \Delta x$



Euler's Method:

We want to estimate $y(x_{final})$ given the IVP $y' = f(x, y), \quad y(x_0) = y_0$

Process: You'll be given Δx

1. Locate a **point** (start at the IC, (x_0, y_0)).
2. Find the **slope** of the solution curve at that point, using $y' = \dfrac{dy}{dx} = slope$
3. Find Δy: $\Delta y = m \cdot \Delta x$
4. **Project** forward to a new point, using the fact that $x_{k+1} = x_k + \Delta x$ and $y_{k+1} = y_k + \Delta y$
5. Continue this process until you reach the target x-value.

- Each iteration of this process is called a "**step**".
- The Δx value is called the "step-size". (The variable h is more commonly used for step-size.)
- The number of steps needed is given by $\dfrac{x_{final} - x_{initial}}{\Delta x}$

Step	**Point**		m = **Slope**	Change in y	**Project** to New Point	
k	x_k	y_k	$y'(x) = f(x_k, y_k)$	$\Delta y = m \cdot \Delta x$	$x_{k+1} = x_k + \Delta x$	$y_{k+1} = y_k + \Delta y$
0	$(x_0$	$y_0)$ → plug in →		find this	$x_1 = x_0 + \Delta x$	$y_1 = y_0 + \Delta y$
1	$(x_1$	$y_1)$ → continue				

Example: Use Euler's method, by hand, to estimate the solution to the IVP $y' = x - y - 2$, $y(0) = 1$ at $y(2)$ using the following step sizes: , and $\Delta x = .5$ Illustrate the polygonal approximation on the slope field.

$\Delta x = 1$ — How many steps? | $n = \frac{x_{final} - x_{initial}}{\Delta x} = \frac{2-0}{1} = 2$ steps

Step	Point		Slope	Change in y	Project to New Point	
k	x_k	y_k	$y' = x - y - 2$	$\Delta y = m \cdot \Delta x$	$x_k + \Delta x$	$y_k + \Delta y$
0	(0	1)	$y' = 0-1-2 = -3$	$\Delta y = -3(1) = -3$	$0+1=1$	$1+(-3) = -2$
1	(1	−2)	$y' = 1-(-2)-2 = 1$	$\Delta y = 1(1) = 1$	$1+1=2$	$-2+(1) = -1$
2	(2	−1)				

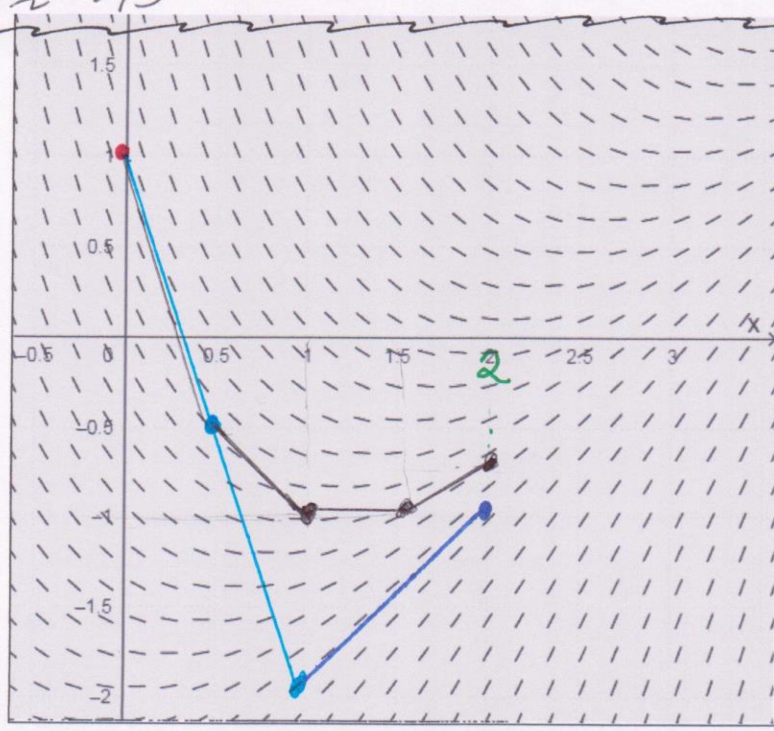
→ 2 steps!

$y(2) \approx -1$ — Euler's Estimate

$\Delta x = .5$ — How many steps? $n = \frac{2-0}{.5} = 4$ steps

Step	Point		Slope	Change in y	Project to New Point	
k	x_k	y_k	$y'(x)$ $x - y - 2$	$\Delta y = m \cdot \Delta x$	$x_k + \Delta x$	$y_k + \Delta y$
0	(0	1)	$0-1-2=-3$	$-3(.5) = -1.5$	$0+.5=.5$	$1+(-1.5) = -.5$
1	(.5	−.5)	$.5+.5-2=-1$	$-1(.5) = -.5$	$.5+.5 = 1$	$-.5+(-.5) = -1$
2	(1	−1)	$1+1-2=0$	$0(.5) = 0$	$1+.5 = 1.5$	$-1+0 = -1$
3	1.5	−1	$1.5+1-2 = +.5$	$+.5(.5) = +.25$	$1.5+.5 = 2$	$-1+(+.25) = -.75$
4	2	−.75				

$y(2) \approx -.75$



Now use the GeoGebra program to confirm your answers above and to continue the process $\Delta x = .1$ $\Delta x = .05$

See link

Error Analysis

Put all of your estimations in the table below.

Then determine the actual value of y(2) using the fact that the analytical solution to the IVP is $\boxed{y = x + 4e^{-x} - 3}$*

**Note: Wolfram will solve the IVP for you. Use the command "Solve" followed by the IVP*

$$y(2) = 2 + 4e^{-2} - 3$$
$$= -.4587 \text{ (to 4 decimal places)}$$

Finally, determine the error. Remember: Error = Estimate – Actual

Δx	Euler's Estimate y(2)	Actual value of y(2)	Error
1	−1	−.4587	−.5413
.5	−.75	−.4587	−.2913
.1	−.5137	−.4587	−.0550
.25	−.5995	−.4587	−.1408

Note: Absolute error would be positive.
The negative tells us this approximation is an underestimate, which is clear from the curve shown in Geogebra.

If the step size is reduced ~~by a factor of 10~~, what effect does this appear to have on the error?

From $\Delta x = .5$ to $\Delta x = .25$, step size reduced by factor of 2

$$\frac{\text{Error}_{.5}}{\text{Error}_{.25}} = \frac{-.2913}{-.1408} = 2.07 \approx 2$$

Error seems to decrease by the same factor as the step size (you can check the other ratios as well. They hold to this pattern, roughly, except $\Delta x = 1$ which is a pretty ~~gross~~ far-off estimate.)

How efficient is this process?

Not very - error goes down linearly.
In Math 287 you'll learn modifications of this method that vastly improve the efficiency.