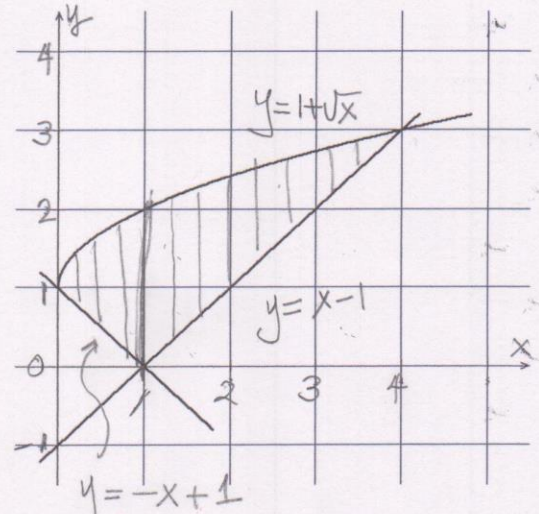
Math 265B: Final Exam **Name:** KEY

(125 pts)

Important note: Since many of these problems are possible to solve using technology, be sure that you are showing ***clear and complete work****. Credit is based on the amount of correct work shown, not just the final answer, and correct answers without the correct work to support them will receive no credit at all. Give exact answers (leave pi as pi, etc.) unless an approximation is asked for.*

1. (4 pts) Consider the region bounded by $y=1+\sqrt{x}$, $y=x-1$, and $y=-x+1$, as shown. Use calculus to find the area of the region. Express the answer as a fraction, not a decimal.



$$A=\int_0^1 (1+\sqrt{x})-(-x+1)\,dx+\int_1^4 (1+\sqrt{x})-(x-1)\,dx$$

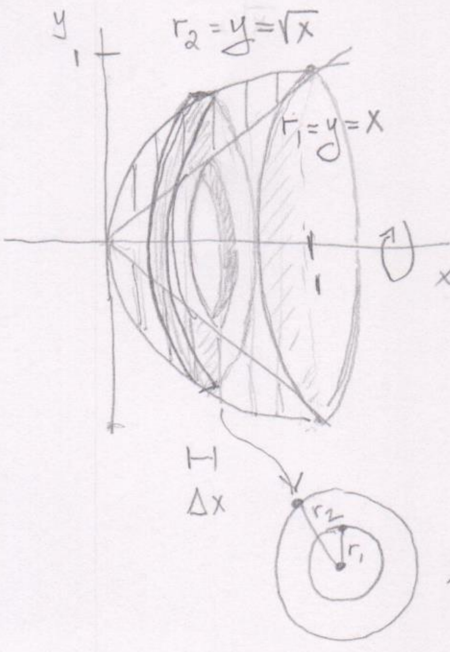
$$=\int_0^1 x^{1/2}+x\,dx+\int_1^4 x^{1/2}-x+2\,dx$$

$$=\left(\frac{2}{3}x^{3/2}+\frac{1}{2}x^2\right|_0^1+\left(\frac{2}{3}x^{3/2}-\frac{1}{2}x^2+2x\right|_1^4$$

$$=\frac{2}{3}+\frac{1}{2}-(0+0)+\left(\frac{2}{3}\cdot 4^{3/2}-\frac{1}{2}4^2+2(4)-\left[\frac{2}{3}-\frac{1}{2}+2\right]\right)$$

$$=\boxed{4\tfrac{1}{3}\text{ or }\frac{13}{3}\text{ unit}^2}$$

2. (6 pts) Find the volume of the solid that is formed by revolving the region bounded by $y=\sqrt{x}$ and $y=x$ about the x-axis. Include a sketch to show your reasoning.



$$V\approx\sum_{\text{slices}}\text{Area of cross section}\cdot\text{thickness}$$

$$=\int_0^1 \pi\left[\sqrt{x}\right]^2-\pi\left[x\right]^2\,dx$$

$$=\pi\int_0^1 x-x^2\,dx$$

$$=\pi\left[\frac{1}{2}x-\frac{1}{3}x^3\right|_0^1=\pi\left[\frac{1}{2}-\frac{1}{3}-(0-0)\right]$$

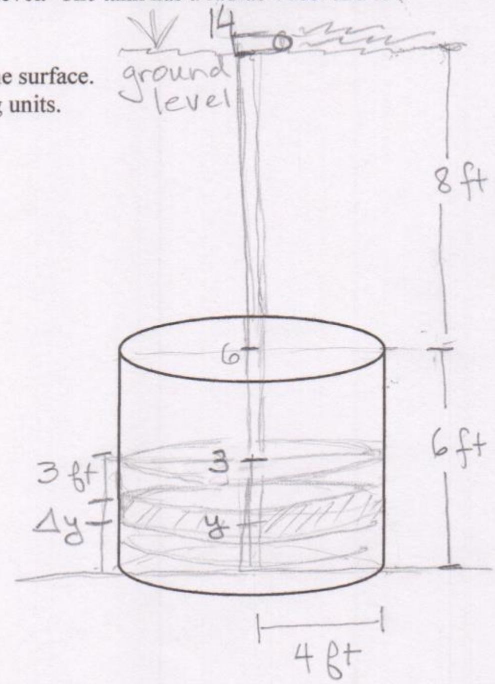
$$=\boxed{\frac{\pi}{6}\text{ unit}^3}$$

Area $=\pi r_2^2-\pi r_1^2$

$r_1=x$, $r_2=\sqrt{x}$

3. (6 pts) A buried cylindrical tank (see picture) has its top 8 feet beneath ground level. The tank has a radius 4 feet and is 6 feet high and is half-full of oil. (Oil weighs 50 lbs/ft^3.)

Use calculus to find the work done against gravity in pumping all of the oil to the surface. Express your answer in terms of ρ, g, π, then give an approximation including units.



$$\text{Work} = \text{Force} \cdot \text{distance}$$
$$\approx \sum (mg)_{slice} \cdot (14-y)$$
$$= g \sum \rho \cdot V_{slice} \cdot (14-y)$$
$$= \rho g \sum A_{slice} \cdot \Delta y \, (14-y)$$

$$\text{Work} = \rho g \int_0^3 \pi (4)^2 (14-y)\,dy$$
$$= 16\rho g \pi \left[14y - \tfrac{1}{2}y^2\right]_0^3$$
$$= 16\rho g \pi \left[14(3) - \tfrac{1}{2}(3)^2 - (0-0)\right]$$
$$= 16\rho g \pi (37.5)$$
$$= \boxed{600\rho g \pi \text{ ft-lbs}} = 600(50)(32)(3.141592654) = \boxed{3{,}015{,}929} \approx 3{,}000{,}000 \text{ ft-lbs}$$

sig figs.

4. (5 pts) Determine which of the integration techniques listed below would be the ***BEST*** to apply to find the following integrals. You do not have to integrate.

Techniques: ***Partial Fraction Decomposition,*** ***Integration by Parts,*** ***u-Substitution ,***

Trig Substitution, Half Angle Trig Identity (Power Reduction), Pythagorean Trig Identify

a.) $\int x\sin(x)\,dx$ Best Technique: Integration by Parts

b.) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ Best Technique: u-sub

c.) $\int \frac{1}{x^3+5x}\,dx$ Best Technique: Partial Fractions

d.) $\int \cos^2(x)\sin^2(x)dx$ Best Technique: Half-Angle Trig ID's

e.) $\int \frac{1}{\sqrt{x^2-25}}dx$ Best Technique: Trig Substitution

5. (5 pts) Integrate. Show complete work for credit, and simplify as much as possible

$\int_1^2 x^2 \ln(x)\, dx$.

LIATE $\Rightarrow$ $u = \ln x$ $\quad dv = x^2$

$du = \frac{1}{x}$ $\quad v = \frac{1}{3}x^3$

$$= \frac{1}{3}x^3 \ln x - \int_1^2 \frac{1}{3}x^3 \cdot \left(\frac{1}{x}\right) dx$$

$$= \frac{1}{3}x^3 \ln x - \int_1^2 \frac{1}{3}x^2\, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 \Big|_1^2$$

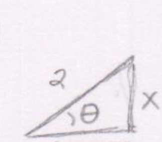
$$= \frac{1}{3}(2)^3 \ln 2 - \frac{1}{9}(2)^3 - \left[\cancel{\frac{1}{3}(1)^3 \ln(1)}^{\,0} - \frac{1}{9}(1)^3\right] = \boxed{\frac{8}{3}\ln 2 - \frac{7}{9}}$$

6. (7 pts) Integrate using Trig Substitution. $\int_1^2 \frac{\sqrt{4-x^2}}{x^2} dx$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$



Note: you can convert the endpoints to θ here or wait and go back to x's at the end.

$x = 1 \Rightarrow 1 = 2\sin\theta$, $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}$

$x = 2$, $2 = 2\sin\theta$, $1 = \sin\theta$, $\theta = \frac{\pi}{2}$

$$= \int_{x=1}^{x=2} \frac{\sqrt{4 - 4\sin^2\theta}}{4\sin^2\theta} \cdot 2\cos\theta\, d\theta$$

$$= \int_{x=1}^{x=2} \frac{\cancel{2}\sqrt{1-\sin^2\theta}}{\cancel{4}\sin^2\theta} \cdot \cancel{2}\cos\theta\, d\theta$$

$$= \int_{x=1}^{x=2} \frac{\cos\theta \cdot \cos\theta}{\sin^2\theta}\, d\theta$$

$$= \int_{x=1}^{x=2} \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int_{x=1}^{x=2} \frac{1-\sin^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int_{x=1}^{2} \csc^2\theta - 1\, d\theta$$

$$= -\cot\theta - \theta \Big|_{x=1}^{2}$$

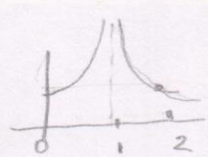
$$= -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right)\Big|_{x=1}^{2}$$

$$= -\frac{\sqrt{0}}{2} - \sin^{-1}(1) - \left(-\frac{\sqrt{3}}{1} - \sin^{-1}\left(\frac{1}{2}\right)\right)$$

$$= \sqrt{3} - \frac{\pi}{2} + \frac{\pi}{6}$$

$$\boxed{= \sqrt{3} - \frac{\pi}{3}}$$

7. (6 points) The given integral is improper. What makes it improper?



$$\int_0^2 \frac{1}{(x-1)^{4/3}}\,dx$$

x = 1 is in the interval [0,2], which is an issue since it causes division by 0.

Evaluate the integral and state whether it converges or diverges. Use proper limit notation and include a graph to illustrate your work.

$$\int_0^2 \frac{1}{(x-1)^{4/3}}dx = \lim_{b\to 1^-}\int_0^b \frac{1}{(x-1)^{4/3}}dx + \lim_{b\to 1^+}\int_b^2 \frac{1}{(x-1)^{4/3}}dx$$

$$\int \frac{1}{(x-1)^{4/3}}dx = \int (x-1)^{-\frac{4}{3}}dx = -3(x-1)^{-\frac{1}{3}} = \frac{-3}{(x-1)^{1/3}}$$

$$= \lim_{b\to 1^-} \frac{-3}{(x-1)^{1/3}}\Big|_0^b + \lim_{b\to 1^+}\frac{-3}{(x-1)^{1/3}}\Big|_b^2$$

$$= \lim_{b\to 1^-}\left[\frac{-3}{(b-1)^{1/3}} - 3\right] + \lim_{b\to 1^+}\left[-3 + \frac{3}{(b-1)^{1/3}}\right]$$

$$= \infty + \infty \Rightarrow \boxed{\text{DIVERGES}}$$

8. (6 pts) (a) Evaluate the integral and state whether it converges or diverges. Use proper limit notation.

$$\int_\pi^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_\pi^b x^{-2}dx =$$

$$= \lim_{b\to\infty} -1x^{-1}\Big|_\pi^b$$

$$= \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{\pi} = 0 + \frac{1}{\pi} = \boxed{\frac{1}{\pi}}$$

$\boxed{\text{CONVERGES}}$

(b) Based on the result from part (a), and the given graph $\frac{1}{x^2}$ and $\frac{\sin^2(x)}{x^2}$,

explain how you can tell whether the integral $\int_\pi^\infty \frac{\sin^2(x)}{x^2}dx$

converges or diverges.

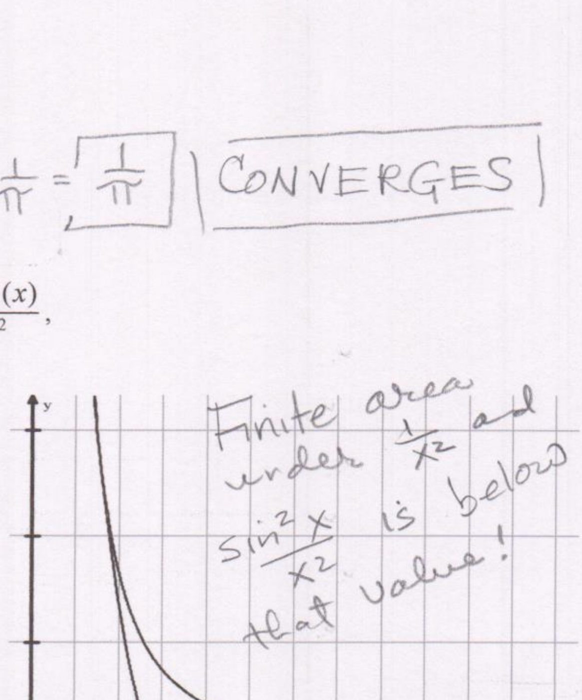
Since $0 \le \frac{\sin^2(x)}{x^2} \le \frac{1}{x^2}$

we know that $\int_\pi^\infty \frac{\sin^2(x)}{x^2}dx$

must be less than $\int_\pi^\infty \frac{1}{x^2}dx$

which converges (is finite)

so $\int \frac{\sin^2(x)}{x^2}dx$ must also $\boxed{\text{converge.}}$



9. (4 pts) If $f(x)$ is positive, increasing, and concave down on [a,b] then which of these is a possible ordering of the approximations for $\int_a^b f(x)dx$? <u>Sketch a graph of the integral and the approximations to explain your choice, using n = 2</u>. A bigger picture is easiest to reason with.

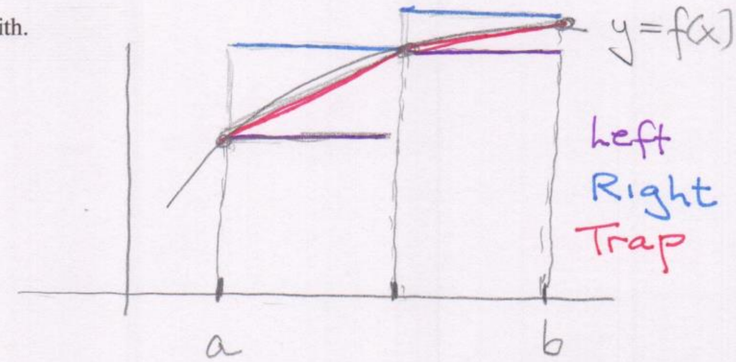
a. LEFT(n) ≤ TRAP(n) ≤ RIGHT(n)

b. RIGHT(n) ≤ TRAP(n) ≤ LEFT(n)

c. RIGHT(n) ≤ LEFT(n) ≤ TRAP(n)

d. LEFT(n) ≤ RIGHT(n) ≤ TRAP(n)

Picture:



10. (2 pts) Determine whether each of the following series converges or diverges. Give a brief reason for each. (Do not do any formal testing here.)

(a) $\sum_{k=1}^{\infty} \left(\frac{3}{5}\right)^k$ converges / diverges (circle one) because $|r| = \frac{3}{5} < 1$ (Geometric Series Test)

(b) $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ converges / diverges (circle one) because $p = \frac{1}{2} < 1$ (p-Series Test)

11. (6 pts) Find the interval of convergence for this series. <u>Include</u> determining whether or not to include the <u>endpoints</u> of the interval. $\sum_{k=0}^{\infty} \frac{x^k}{2^k}$

Ratio Test Method

$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$ for convergence

$\lim_{k\to\infty} \left|\frac{x^{k+1}}{2^{k+1}} \cdot \frac{2^k}{x^k}\right| < 1$

$\lim_{k\to\infty} \left|\frac{x}{2}\right| < 1$

$|x| \cdot \lim_{k\to\infty} \frac{1}{2} < 1$

$|x| \cdot \frac{1}{2} < 1$

$|x| < 2$

$-2 < x < 2$

Endpt Test

$x = -2$

Does $\sum_{k=0}^{\infty} \frac{(-2)^k}{2^k}$ converge? nope

$= \sum_{k=0}^{\infty} (-1)^k$ diverges

$x = 2$

Does $\sum_{k=0}^{\infty} \frac{2^k}{2^k}$ converge? nope

$= \sum_{k=0}^{\infty} 1$ diverges

⇒ Final interval of convergence $\boxed{(-2,2)}$

12. (3 pts) Derive the first three non-zero terms of the Taylor Series formula for cos(x). Centered at $x=0$

$f(x) = \cos x \qquad f(0) = 1$

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f'''(x) = \sin x \qquad f'''(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

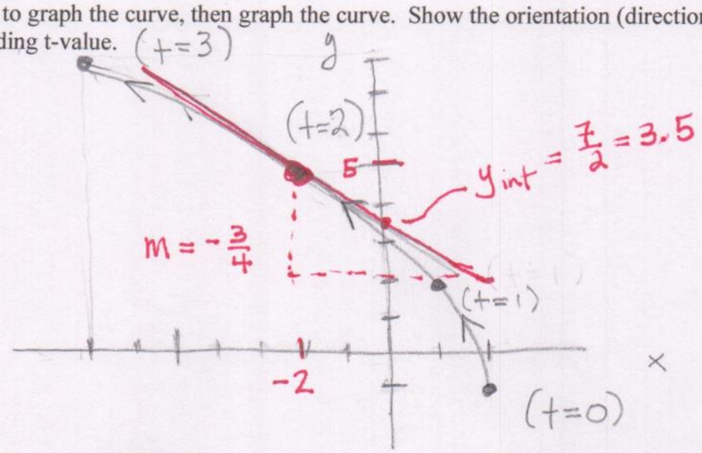
$$\cos x \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 1 + 0x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 = \boxed{1 - \frac{x^2}{2!} + \frac{x^4}{4!}}$$

13. (9 pts) Suppose a particle is moving along the path described by the parameterization $\begin{cases} x = 2 - t^2 \\ y = 3t - 1 \end{cases} \quad 0 \le t \le 3$

(a) Set up and fill in an appropriate table to graph the curve, then graph the curve. Show the orientation (direction of travel) and label each point with it corresponding t-value.

t	x	y
0	2	−1
1	1	2
2	−2	5
3	−7	8



(b) Find the equation of the tangent line at the point where t = 2. Sketch the line on your graph from part (a).

$y - y_2 = m(x - x_2)$

$t = 2 \Rightarrow x_2 = -2,\ y_2 = 5$

$m_{tan} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{-2t}$

$y - 5 = -\frac{3}{4}(x+2)$

$y - 5 = -\frac{3}{4}x - \frac{3}{2}$

$m_{tan}\Big|_{t=2} = \frac{3}{-2(2)} = -\frac{3}{4} = \frac{\Delta y}{\Delta x}$

$\boxed{y = -\frac{3}{4}x + \frac{7}{2}}$

(Extra credit (2 pts): Find the parametric form of the tangent line equation, including the appropriate time shift to ensure the particle is at the point of tangency at the right time.)

$x = x_2 + \Delta x(t-2)$

$\boxed{x = -2 - 4(t-2)}$ (best)

OR $x = -4t + 6$ (ok)

$y = y_2 + \Delta y(t-2)$

$\boxed{y = 5 + 3(t-2)}$ (best)

OR $y = 3t - 1$ (ok)

Domain: $t \ge 2$

14. (6 pts) Find a parameterization of each of the following curves. Be sure to include the domain! $\boxed{a \le t \le b}$

(a) The line segment that begins at point P(2, -1) and ends and point Q(-1,3) $\Longrightarrow \Delta x = -1-2 = -3$

$\Delta y = 3+1 = 4$

$x = x_0 + \Delta x\, t$

$x = 2 - 3t$

$y = y_0 + \Delta t$

$y = -1 + 4t$

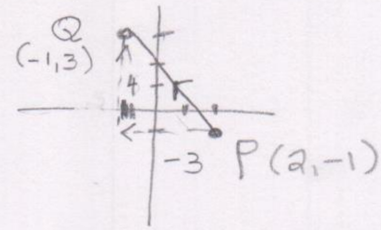
$\boxed{\begin{array}{l} x = 2-3t \\ y = -1+5t \end{array}}$ $\boxed{0 \le t \le 1}$

check:

$t=0 \quad x=2,\ y=-1 \Rightarrow P$

$t=1 \quad x=-1,\ y=3 \Rightarrow Q$

yes.



(b) A circle of radius 4, oriented counterclockwise, with center at the point (1, 3) and period of 6 seconds.

Circle:

$x = R\cos(bt) + h$

$y = R\sin(bt) + k$

$\text{period} = \frac{2\pi}{b}$

$6 = \frac{2\pi}{b}$

$b = \frac{\pi}{3}$

$R = 4$

$(h,k) = (1,3)$

$\boxed{\begin{array}{l} x = 4\cos\left(\frac{\pi}{3}t\right) + 1 \\ y = 4\sin\left(\frac{\pi}{3}t\right) + 3 \end{array}}$ $\boxed{0 \le t \le 6}$

15. (6 pts) Convert each of the xy-equations into a polar equation. Solve for r, if possible.

(a) $x^2 + y^2 = 6y$

$r^2 = 6r\sin\theta$

$\boxed{r = 6\sin\theta}$

(b) $y = 5x$

$r\sin\theta = 5r\cos\theta$

ok $\boxed{\sin\theta = 5\cos\theta}$

better $\boxed{\tan\theta = 5}$

best $\boxed{\theta = \tan^{-1}(5)}$

(c) $y = 5$

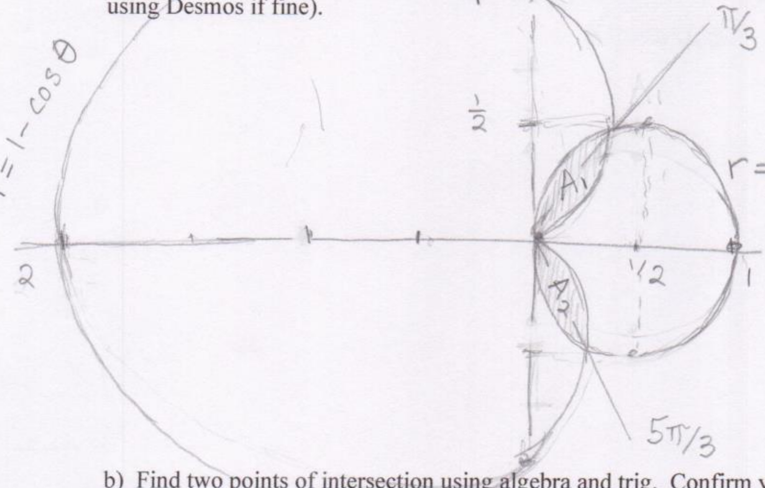
$r\sin\theta = 5$

$r = \frac{5}{\sin\theta}$

$\boxed{r = 5\csc\theta}$

16. (14 pts) Consider the region, defined using polar coordinates, where the area inside the curve $r = \cos\theta$ overlaps the area inside curve $r = 1 - \cos\theta$ (so the area that is common to both).

a) Sketch a graph of the curves and shade the region as described. You don't have to show work for the graphs (i.e., using Desmos if fine).

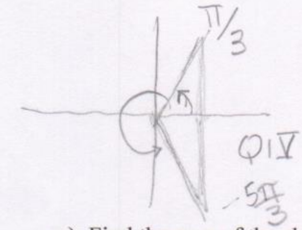


b) Find two points of intersection using algebra and trig. Confirm you work and find the third point using a graph. (Extra credit (2 pts): Explain why the third point can't be found algebraically.

Intersection where $\cos\theta = 1 - \cos\theta$

$2\cos\theta = 1$

QI $\quad \cos\theta = \frac{1}{2}$



$\theta \in QI, QIV$ where $\cos\theta > 0$

$\theta = \frac{\pi}{3}$ $\quad$ $\theta = \frac{5\pi}{3}$

$r = \cos\frac{\pi}{3} = \frac{1}{2}$ $\quad$ $r = \cos\frac{5\pi}{3} = \frac{1}{2}$

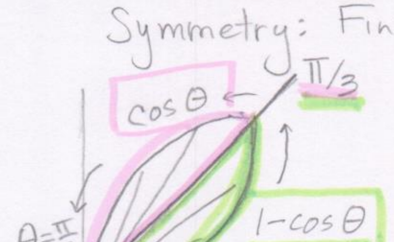
$\boxed{(r,\theta) = \left(\frac{1}{2}, \frac{\pi}{3}\right)}$ $\quad$ $\boxed{(r,\theta) = \left(\frac{1}{2}, \frac{5\pi}{3}\right)}$

By inspection: $(x,y) = (0,0)$ is also a point of intersection

XC: 2 particles traveling $r = \cos t$ and $r = 1 - \cos t$ would cross paths at $(0,0)$ but NOT at the same time. Their paths share this point but they would not collide at that point. They would collide at the other 2 points!

c) Find the area of the shaded region, using calculus.

Symmetry: Find A_1 and double.



A_1: $\int_0^{\pi/3} \frac{1}{2}(1-\cos\theta)^2\, d\theta = \frac{\pi}{4} - \frac{7\sqrt{3}}{16}$ *

$+ \int_{\pi/3}^{\pi/2} \frac{1}{2}(\cos\theta)^2\, d\theta = \frac{\pi}{24} - \frac{\sqrt{3}}{16}$ *

$\frac{7\pi}{24} - \frac{8\sqrt{3}}{16}$

so $A_1 = \frac{7\pi}{24} - \frac{\sqrt{3}}{2}$

thus, total area is

$A = 2A_1 = 2\left(\frac{7\pi}{24} - \frac{\sqrt{3}}{2}\right)$

$\boxed{= \frac{7\pi}{12} - \sqrt{3}}$

* see next page

16c

$$* \int_0^{\pi/3} \frac{1}{2}(1-\cos\theta)^2 \, d\theta \longrightarrow \frac{1}{2}(1-\cos\theta)^2$$

$$= \frac{1}{2}(1-2\cos\theta+\cos^2\theta)$$

$$= \frac{1}{2}\left(1-2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right)$$

$$= \frac{3}{4} - \cos\theta + \frac{1}{4}\cos 2\theta$$

$$= \int_0^{\pi/3} \frac{3}{4} - \cos\theta + \frac{1}{4}\cos 2\theta \, d\theta$$

$$= \frac{3}{4}\theta - \sin\theta + \frac{1}{8}\sin 2\theta \Big|_0^{\pi/3}$$

$$= \frac{3}{4}\left(\frac{\pi}{3}\right) - \sin\frac{\pi}{3} + \frac{1}{8}\sin\frac{2\pi}{3} - (0 - \cancel{2\sin 0}^{\,0} + \cancel{\tfrac{1}{4}\sin 0}^{\,0})$$

$$= \frac{\pi}{4} - \frac{\sqrt{3}}{2} + \frac{1}{8}\frac{\sqrt{3}}{2} = \boxed{\frac{\pi}{4} - \frac{7\sqrt{3}}{16}}$$

$$* \int_{\pi/3}^{\pi/2} \frac{1}{2}(\cos\theta)^2 \, d\theta \longrightarrow \frac{1}{2}\cos^2\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}(1+\cos 2\theta)\right)$$

$$= \int_{\pi/3}^{\pi/2} \frac{1}{4}(1+\cos 2\theta) \, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta \Big|_{\pi/3}^{\pi/2}\right.$$

$$= \frac{1}{4}\left(\left[\frac{\pi}{2} + \cancel{\tfrac{1}{2}\sin\pi}^{\,0}\right] - \left[\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right]\right)$$

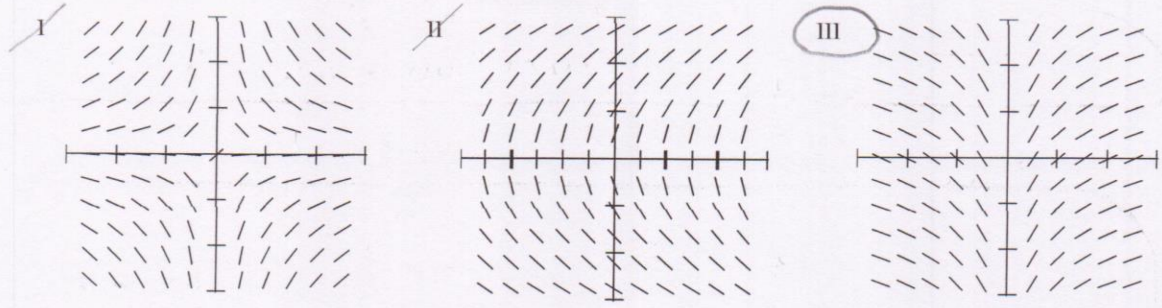
$$= \frac{1}{4}\left[\frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2}\right]$$

$$= \boxed{\frac{\pi}{24} - \frac{\sqrt{3}}{16}}$$

17. (3 pts) Determine which slope field matches the given differential equation. The window is [-3,3] by [-3, 3] in each graph.

$x=0 \Rightarrow$ Vertical tan. lines
$x>0 \Rightarrow$ Positive slope
$x<0 \Rightarrow$ Negative slope

$\frac{dy}{dx}=\frac{1}{x}$ Answer: III



18. (6 pts) The Initial Value Problem given below is an example of the type of DE's you see $\boxed{\text{in logistic models.}}$ Find the solution to the IVP in explicit form.

$\frac{dy}{dt}=\frac{1}{y(5-y)} \quad y(0)=2.$

Solution

$$\boxed{(-6)\; -\frac{1}{3}y^3+\frac{5}{2}y^2=t+\frac{22}{3}\;(-6)}$$

$$\boxed{\text{OR } 2y^3-15y^2=-6t-44}$$

No!
$\frac{dy}{dt}=y(5-y)$ for logistic!

$\int y(5-y)\,dy=\int 1\,dt$

$\int 5y-y^2\,dy=t+C$

$\frac{5}{2}y^2-\frac{1}{3}y^3=t+C$

$t=0 \Rightarrow y=2$

$\frac{5}{2}(2)^2-\frac{1}{3}(2)^3=0+C$

$C=10-\frac{8}{3}=\frac{22}{3}$

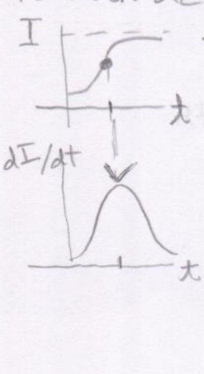
Explicit solution - see Wolfram

(This became kind of a dumb problem. Solns to cubic equations are really ugly and not particularly useful.)

Much more interesting to look at the curve and slope field and how Euler's method FAILS massively! See attached...

Extra credit (3 pts): Use calculus to find the inflection point of the solution curve. Why is the inflection point important for logistic models of real situations, like epidemics? For an actual logistic model

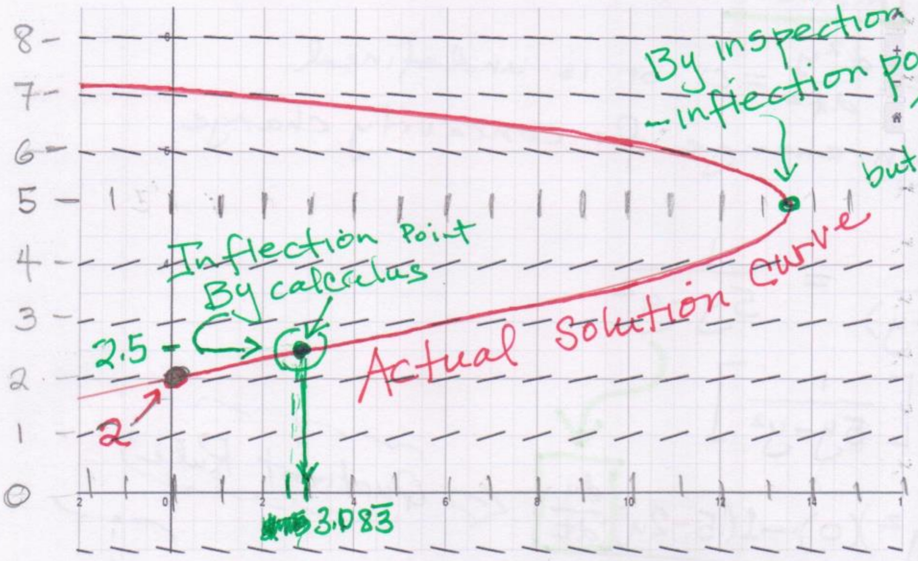
See next page



the point of inflection indicates where the rate of increase "backs off". For an epidemic, it corresponds to the peak in the infection rate curve.

#18 More interesting considerations AND <u>extra credit inflection analysis</u>

This graph shows the slope field for $dy/dt = \frac{1}{y(5-y)}$ with the red solution curve for the IVP, with $\boxed{\text{IC } y(0) = 2.}$



Note the vertical tangent lines at $y=0$ and $y=5$

This graph shows the Euler's approximation for the solution curve ($\Delta x = .1$, so the curve looks smooth since the segments are so small.)

Note how the "curve" goes bananas once it reaches the vertical tangent point and gets bounced around back and forth.

This illustrates that approximation methods aren't foolproof!

#18 Extra credit

Strategy to find inflection point:

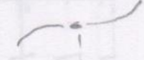
1. Find second derivative
   — use implicit differentiation
2. Find where $\frac{d^2y}{dt^2} = 0$ or is undefined
3. Check sign change for concavity change

$$\frac{dy}{dt} = \frac{1}{y(5-y)} = \frac{1}{5y-y^2}$$

$$\frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{d}{dt}\left[\frac{1}{5y-y^2}\right]$$

$$\frac{d^2y}{dt^2} = \frac{(5y-y^2)(0) - 1(5-2y)\boxed{\frac{dy}{dt}}}{(5y-y^2)^2}$$

⇐ Quotient Rule



$\frac{d^2y}{dt^2}$ − 0 +

2.5

$$= \frac{2y-5}{(5y-y^2)^2} \cdot \boxed{\frac{1}{(5y-y^2)}} = \frac{2y-5}{(5y-y^2)^3} = 0 \text{ for } y = \frac{5}{2} = 2.5$$

→ undefined for $y=0,\ y=5$

$y = 2.5$ in $2y^3 - 15y^2 = -6t - 44$

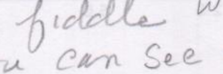
$$2(2.5)^3 - 15(2.5)^2 + 44 = -6t$$
$$-18.5 = -6t$$
$$t = +3.08\overline{3}$$

Similarly:
$y = 0 \Rightarrow t = -7.\overline{3}$
$y = 5 \Rightarrow t = 13.5$

We can ignore these since the curve ends at these points due to turning back, thus failing VLT.

Inflection point on solution function curve (remember VLT!) is at $\boxed{(3.08\overline{3},\ 2.5)}$

If you graph the solution curve and fiddle with the window, you can see the concavity change. →



11. (14 pts) A cup of coffee cools at a rate proportional to the difference between the coffee's temperature, H, and the ambient temperature, A. Suppose you're outside where the temperature is $60^{\text{o}} F$ with a freshly-made cup of coffee that is $170^{\text{o}} F$.

(a) Set up the differential equation and initial condition that model this situation.

$A = 60°F$
$H_0 = 170°F$
$H =$ Temp at time, t (°F)
$t =$ time

$$(*) \quad \frac{dH}{dt} = k(60 - H), \quad H(0) = 170$$

(b) Is the differential equation Autonomous? How can you tell?

yes, t is not a variable in the rate (slope) function.

(c) Find the equlibrium solution (show work) using your DE and sketch a slope field that shows the equilibrium solution and the solution curve for the IC you found in part (a).

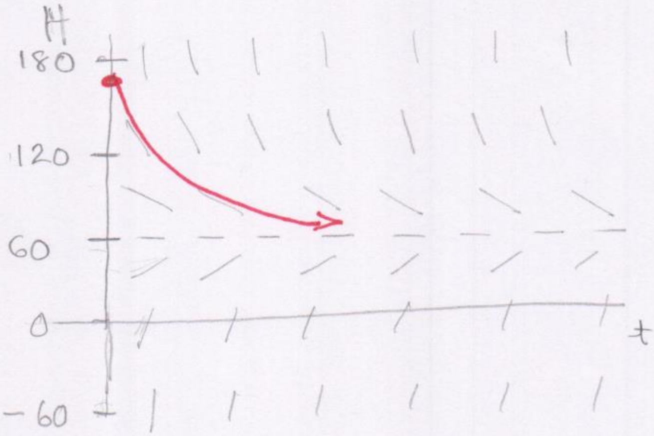
$$\frac{dH}{dt} = k(60 - H) = 0$$

for $H = 60$

(*) Note: You can set this up either format

ie $\frac{dH}{dt} = k(H - 60)$ is fine,

but k will be negative.



(d) Solve the IVP using Separation of Variables (show all steps).

$$\frac{dH}{dt} = k(60 - H)$$

$$\frac{dH}{dt} = -k(H - 60)$$

$$\int \frac{dH}{H - 60} = \int -k \, dt$$

$$\ln|H - 60| = -kt + C$$

$$|H - 60| = e^{-kt + C}$$

$$H - 60 = \pm e^{C} \cdot e^{-kt}$$

$$H - 60 = C_1 e^{-kt}$$

Find C_1: $H(0) = 170$

$$170 - 60 = C_1 e^{0}$$

$$C_1 = 110$$

$$H - 60 = 110 e^{-kt}$$

$$\boxed{H = 60 + 110 e^{-kt}}$$

(#11 continued)

(e) If the coffee cooled to be $115^{O}F$ after 5 minutes, what is the value of k? (Note: this will take some Precalculus algebra to solve. See Section 8.5 [9.5] for an example showing this.)

$t = 5$
$H = 115$

$H = 60 + 110e^{-kt}$

$115 = 60 + 110e^{-k(5)}$

$\frac{55}{110} = \frac{110e^{-5k}}{110}$

note: this is a percent of surplus heat lost.

$\frac{1}{2} = e^{-5k}$

$\ln(\frac{1}{2}) = \ln e^{-5k}$

$-5k = \ln(\frac{1}{2})$

$k = \frac{\ln(\frac{1}{2})}{-5\,min}$ OR $\frac{\ln 2}{5}$

$k \approx .139$

(Extra credit (2 pts): Explain what the physical meaning is of k, including units, in the context of this problem.

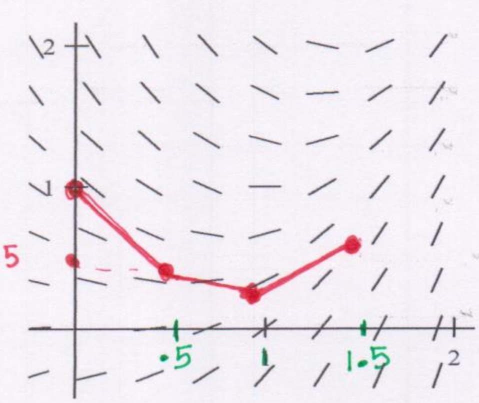
The units of k are $\frac{1}{min}$ which suggests k is a rate. The coefficient of e^{-kt} is 110° which is the difference between the original temp and the ambient temp. So k is the rate at which this surplus of heat is lost, per minute, expressed as percent (percent loss, per minute)

7. (7 pts) Consider the I.V.P. $\frac{dy}{dx} = x^2 - y, \quad y(0) = 1$

(a) Use Euler's Method, with $\Delta x = 0.5$ to estimate the value of $y(x)$, for $x = 1.5$ Clearly show all of your work.

number of steps: $n = \frac{b-a}{\Delta x} = \frac{1.5-0}{.5} = 3$

k	x_k	y_k	$\frac{dy}{dx} = x_k^2 - y_k$	$\Delta y = \frac{dy}{dx}(\Delta x)$
0	0	1	$0^2 - 1 = -1$	$-1 \cdot (.5) = -.5$
1	.5	.5	$(.5)^2 - (.5) = -.25$	$-.25(.5) = -.125$
2	1	.375	$(1)^2 - .375 = .625$	$.625(.5) = .3125$
3	1.5	.6875		



$y(1.5) \cong$ .6875

(b) On the slope field provided, sketch the polygonal Euler's solution $\boxed{\text{"curve"}}$, based on your work from part (a).

.5

as shown