Math 265B Techniques of Integration: Partial Fractions

Goal: Integrate rational functions, i.e, find $\int \frac{P(x)}{Q(x)}dx$

In order to find the anti-derivative of a rational function, we may need to break the larger fraction (rational function) down into smaller components using the technique of "Partial Fraction Decomposition". The steps for doing this are as follows:

Steps:	**Example**: Integrate: $\int \frac{x^3}{x^2-4}dx$
1. If the degree of the numerator is greater than or equal to the degree of the denominator, then use long division to divide the fraction. 2. Factor the denominator of the remaining fraction. (Ignore the non-fractional part entirely for the time being.) 3. For each distinct linear factor, create a component fraction. (See the following guidelines for other types of decomposition). 4. Our goal now is to evaluate the constants in the component fractions. To do this • Clear the fractions (multiply by the LCD) • Since the decomposition must be true no matter what value x takes on, you are free to substitute any convenient value of x that you choose. In this case, let x = the zero of each factor then solve. • You can also use the method of Equating Coefficients in which you simplify the right hand side then set up a system of equations based on the coefficients of each power of x. 5. The fraction is now decomposed. Rewrite the original integrand in decomposed form and integrate.	

Note that there are 4 possible types of decomposition we can encounter:

1. Distinct linear factors:

$$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)} = \frac{A}{(a_1x+b_1)} + \frac{B}{(a_2x+b_2)}$$

Example: $\frac{5x}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

2. Repeated linear factors:

$$\frac{P(x)}{(a_1x+b_1)^n} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_n}{(a_1x+b_1)^n}$$

Example: $\frac{2x^2+1}{(x+4)^3} = \frac{A}{x+4} + \frac{B}{(x+4)^2} + \frac{C}{(x+4)^3}$

3. Irreducible quadratic factors:

$$\frac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_3)} = \frac{A_1x+B_1}{(a_1x^2+b_1x+c_1)} + \frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)}$$

Example: $\frac{1}{(x^2+3)(x+1)} = \frac{Ax+B}{x^2+3} + \frac{C}{x+1}$

4. Repeated irreducible quadratic factors:

$$\frac{P(x)}{(a_1x^2+b_1x+c_1)^2} = \frac{A_1x+B_1}{(a_1x^2+b_1x+c_1)} + \frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2}$$

Example: $\frac{1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

PRACTICE: Evaluate. $\int \frac{x-1}{x^4+x^2}\,dx$