Math 265B: Convergence of Power Series (Chapter 10)

1. Convergence Theorem for Power Series

a) Let $c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots = \sum_{k=0}^{\infty} c_k (x-a)^k$ be any power series**.** Then

(i) the series converges only for $x = a$ (and becomes just c_0) or

(ii) the series converges for all values of x or

(iii) there is a positive number R > 0 such that the series converges for all x for which $|x-a| < R$ and diverges for all x for which $|x-a| > R$.

R is called the "radius of convergence".

The interval of convergence must contain the interval $a - R < x < a + R$.

To completely identify the interval of convergence, it needs to be determined whether the series converges at the endpoints of the interval.

NOTE: Every power series converges for x = a. Some series converge only for x = a. Some series converge for x = a and for all other reals, and some series converge for x =a and on an interval around a.

1. Determine the interval of convergence for the following series "by inspection":

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \sum_{k=0}^{\infty} (4x)^k$$

2. We can use the **Ratio Test** is used to determine the radius and interval of convergence of power series.

Given the series $\sum_{n=1}^{\infty} a_n$, $a_n \neq 0$ and $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ or $\left| \frac{a_{n+1}}{a_n} \right| \to \infty$ as $n \to \infty$.

Then i) if L < 1, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii) If L > 1 or if $\left| \frac{a_{n+1}}{a_n} \right| \to \infty$, the series diverges.

iii) If L = 1, the test is inconclusive.

iv)

Determine the radius of convergence and interval of convergence for

$$\sum_{k=0}^{\infty} \frac{x^k}{4k}$$

$$\sum_{k=1}^{\infty} \frac{2^k}{k} (x-3)^k .$$

3. We can construct new series from known series and find the interval of convergence at the same time:

Find power series for the functions below by using old series. Include the interval of convergence for each.

i) $\dfrac{1}{2x+1}$ ii) $\dfrac{x}{2x+1}$ iii) $\dfrac{1}{x+2}$ iv) e^{x^2}

Derivatives and Integrals of Power Series:

If $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$ has an interval of convergence I, then

i) the derivative of f converges on the interior of I

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 3c_4(x-a)^2 + \ldots = \sum_{n=1}^{\infty} n \cdot c_n(x-a)^{n-1}$$

ii) the integral of f converges on the interior of I

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$