8.4 [9.4]: Special First Order Differential Equations

Summary of key points from Sections 8.1 – 8.3 with t = time as the independent variable.

First Order DE's: These are DE's of the form $y' = f(t, y)$, for example $y' = 2t + y$

Initial Value Problem: This is DE coupled with a specific point $y(t_0) = y_0$ on the solution curve. This specific point is called the **Initial Condition**.

Goal:

Given a First Order DE, find the General Solution, which is the family of all solution curves (functions) that satisfy the Differential Equation.

Given an IVP (DE + IC), find the one specific solution curve (function) in the family that begins at the IC point.

More generally, we can find the entire curve that passes through the IC point, though applications don't usually require you to go backwards in time from the given initial time.

How to find solutions to DE's or IVP's

Slope (Direction) Fields. The **family** of solution curves for a First Order DE can be visualized using a slope field, since we know from Calc 1 that y' is the slope of the tangent to the curve, y, and that y is the solution we're after. If you add an Initial Condition, then you can zero in on the one curve that goes through that point.

Analytical (algebraic) Solutions. The only method we learn in this course is Separation of Variables which applies only to Separable DE's.

Putting it all together:

Example: Given the differential equation (D.E.) $y'(t) = 4 - y$

(a) Sketch the slope field for the D.E. "by hand". Check using the Slope Field Generator.

(b) Sketch the solution curves for the following initial conditions (I.C.'s),

A. y(0) = 1 **B.** y(0) = 4 **C.** y(0) = 5

Again, check your work using the Slope Field Generator

(c) Find the General Solution to the D.E. using Separation of Variables.

Check your work using Wolfram Alpha

(d) Find the solution to the Initial Value Problems (for each of the given I.C.'s)

Check your work using Wolfram

(a) $y' = 4 - y$

Slope is zero when

$y' = 4 - y = 0$

$\Rightarrow y = 4$

Slope is neg when

$y' = 4 - y < 0$

$-y < -4$

$y > 4$

Slope is pos when

$y' = 4 - y > 0$

$-y > -4$

$y < 4$

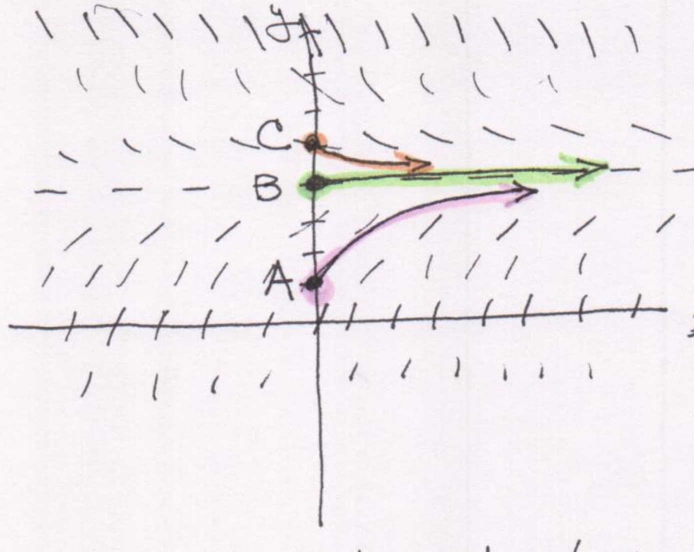
For $y = 0$ $\quad y' = 4 - 0 = 4$

steep!

As $y \to 4^-$ slope gets smaller

As $y \to 4^+$ slope gets smaller

For $y = 8$ $\quad y' = -4 \Rightarrow$ steep!



t	y	$y' = 4 - y$	
doesn't	0	4	smaller
matter	1	3	slope
	2	2	- less
no t	3	1	steep
in slope	4	0	
function	5	-1	getting
	6	-2	steeper
	7	-3	

(b) $y(0) = 1$ $\qquad y(0) = 4$ $\qquad y(0) = 5$

IC's

(c) $y' = 4 - y$

$\frac{dy}{dt} = 4 - y$

$\frac{dy}{4-y} = dt$

TIP: factor out neg

$\frac{dy}{-(y-4)} = dt$

$\int \frac{dy}{y-4} = \int - dt$

Abs value is necessary!

$\ln|y-4| = -t + C$

$e^{\ln|y-4|} = e^{-t+C}$

$|y-4| = e^{-t} \cdot e^{C}$

$y - 4 = \pm e^{C} \cdot e^{-t}$

Call this C_1: $C_1 = \pm e^{C}$

$y - 4 = C_1 e^{-t}$

$y = 4 + C_1 e^{-t}$

(d) IC: $y(0) = 1$

$y(t) = 4 + C_1 e^{-t}$

$1 = 4 + C_1 e^{-(0)}$

$-3 = C_1 \cdot 1$

A $\quad C_1 = -3$

$\Longrightarrow \boxed{y(t) = 4 - 3e^{-t}}$

as $t \to \infty$

$e^{-t} \to 0$

so $y(t) \to 4$

IC $y(0) = 4$

$y(t) = 4 + C_1 e^{-t}$

$4 = 4 + C_1 e^{0}$

B $\quad C_1 = 0$

$\boxed{y(t) = 4}$

(horizontal line
constant function)
called an "Equilibrium
Solution"

IC $y(0) = 5$

$y(t) = 4 + C_1 e^{-t}$

$5 = 4 + C_1 e^{0}$

C $\quad C_1 = 1$

$y(t) = 4 + e^{-t}$

again, as $t \to \infty$

$y(t) \to 4$