Math 265B Taylor Polynomials Homework Assignment

1. Use derivatives to find the Taylor polynomials of degree n approximating the given functions for x near 0.

   (a) $\frac{1}{1+x}$, n = 6

   (b) $\sqrt{1+x}$, n = 4

   (c) $\sqrt[3]{1-x}$, n = 4

   (d) arctan x, n = 5 (Use the Quotient Rule)

   (e) ln (1 + x), n = 7

   (f) $\frac{1}{\sqrt{1+x}}$, n = 4

2. Use derivatives to find the Taylor polynomial of degree n for x near the given point a.

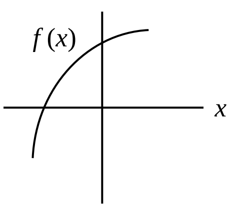
   (a) sin x, a = π/2, n = 6

   (b) cos x, a = π/4, n = 5

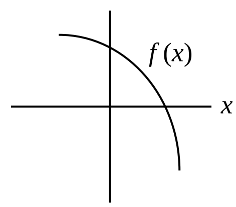
   (c) e^x, a = 1, n = 5

3. For the following problems, suppose $P_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function f about x = 0. What can you say about the signs of a, b, c if f has the graph given below? (Show your understanding of the derivation of the Taylor polynomial as you justify your answers. That is, how are a, b, and c related to f and its derivatives?)

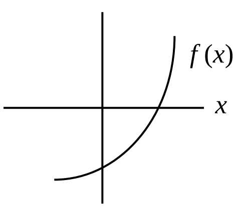
   (a)



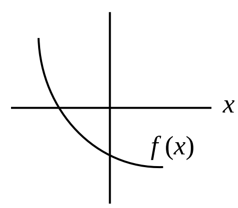
   (b)



   (c)



   (d)



4. Suppose the function $f(x)$ is approximated near x = 0 by a sixth degree Taylor polynomial $P_6(x) = 3x - 4x^3 + 5x^6$. Give the value of each of the following and justify each answer.

   (a) $f(0)$ (b) $f'(0)$ (c) $f'''(0)$ (d) $f^{(5)}(0)$ (e) $f^{(6)}(0)$

5. Suppose g is a function which has continuous derivatives, and that g(5) = 3, g′(5) = -2, g′′(5) = 1, g′′′(5) = -3. What is the Taylor polynomial of degree 3 for g near 5?

6. Find the third-degree Taylor polynomial for $f(x) = x^3 + 7x^2 - 5x + 1$ about x = 0. What do you notice? Make a conjecture about Taylor approximations in the case when f is itself a polynomial.

7. (a) Find the Taylor polynomial approximation of degree 4 about x = 0 for the function $f(x) = e^{x^2}$.

   (b) Compare this result to the Taylor polynomial approximation of degree 2 for the function $f(x) = e^x$ about x = 0. What do you notice?

   (c) Use your observation in part (b) to write out the Taylor polynomial approximation of degree 20 for the function in part (a). Note: You do not need to write out all the intermediate terms. Write out enough to show the pattern and then use the ellipsis and show the 20th term.

   (d) What is the Taylor polynomial approximation of degree 5 for the function $f(x) = e^{-2x}$?

8. Use the Taylor polynomial in 1.(b) to approximate $\sqrt{1.25}$ and $\sqrt{0.81}$. Compare the approximations with the values that your calculator gives you.

Math 265B: Taylor Polynomials Answers:

1. (a) $P_6(x)=1-x+x^2-x^3+x^4-x^5+x^6$ (b) $P_4(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4$

(c) $P_4(x)=1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3-\frac{10}{243}x^4$ (d) $P_5(x)=x-\frac{x^3}{3}+\frac{x^5}{5}$

(e) $P_7(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}-\frac{x^7}{7}$ (f) $P_4(x)=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\frac{35}{128}x^4$

2. (a)
$$P_6(x)=1-\frac{(x-\pi/2)^2}{2!}+\frac{(x-\pi/2)^4}{4!}-\frac{(x-\pi/2)^6}{6!}$$
$$=1-\frac{(2x-\pi)^2}{8}+\frac{(2x-\pi)^4}{384}-\frac{(2x-\pi)^6}{46080}$$

(b)
$$P_5(x)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}(x-\pi/4)}{2}-\frac{\sqrt{2}(x-\pi/4)^2}{4}+\frac{\sqrt{2}(x-\pi/4)^3}{12}+\frac{\sqrt{2}(x-\pi/4)^4}{48}-\frac{\sqrt{2}(x-\pi/4)^5}{240}$$
$$=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}(4x-\pi)}{8}-\frac{\sqrt{2}(4x-\pi)^2}{64}+\frac{\sqrt{2}(4x-\pi)^3}{768}+\frac{\sqrt{2}(x-\pi/4)^4}{12288}-\frac{\sqrt{2}(x-\pi/4)^5}{245760}$$

(c) $P_5(x)=e+e(x-1)+\frac{e}{2}(x-1)^2+\frac{e}{6}(x-1)^3+\frac{e}{24}(x-1)^4+\frac{e}{120}(x-1)^5$

3. For these problems, use the fact that $a = f(0)$, $b = f'(0)$ and $c = f''(0)/2!$. In your justification you need to address whether f(x) increasing or decreasing, being concave up or concave down and how that affects signs. Be complete in your explanations.

(a) a > 0, b > 0, c < 0 (b) a > 0, b < 0, c < 0 (c) a < 0, b > 0, c > 0 (d) a < 0, b < 0, c > 0

4. (a) 0 (b) 3 (c) -24 (d) 0 (e) 3600

5. $P_3(x)=3-2(x-5)+\frac{1}{2}(x-5)^2-\frac{1}{2}(x-5)^3$

6. $P_3(x)=1-5x+7x^2+x^3=f(x)$ exactly. If f(x) is an n-degree polynomial, its Taylor polynomial of the same degree will match it exactly.

7. (a) $P_4(x)=1+x^2+\frac{x^4}{2}$ (b) $P_2(x)=1+x+\frac{x^2}{2}$ You can find the fourth degree Taylor polynomial for e^{x^2} by substituting x^2 for x in the second degree Taylor polynomial for e^x.

(c) $P_{20}(x)=1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\frac{x^{10}}{5!}+\cdots+\frac{x^{20}}{10!}$

(d) Substitute (-2x) for x in the fifth degree Taylor polynomial for e^x.

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$$

$$e^{-2x}=1+(-2x)+\frac{(-2x)^2}{2!}+\frac{(-2x)^3}{3!}+\frac{(-2x)^4}{4!}+\frac{(-2x)^5}{5!}$$
$$=1-2x+2x^2-\frac{4x^3}{3}+\frac{2x^4}{3}-\frac{4x^5}{15}$$

8. For $\sqrt{1.25}$, let x = 0.25. Ans: 1.118011475. Calculator: 1.118033989

For $\sqrt{0.81}$, let x = -0.19 Ans: 0.9000079059 Calculator: 0.9