Math 265B: Test 1 **(Wright, S20)** **Name:** KEY

In class: ________/85 points Take Home:________/15 points

Credit is based on correct work that shows the use of calculus. Use of specified method (where applicable). Simplify all answers as much as possible unless otherwise indicated. Only scientific calculators may be used on this exam.

1. (22 pts) Find each of the following:

3 (a) $\int \frac{2x+1}{x^2+x}dx = \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|x^2+x| + C}$

$u = x^2 + x$
$du = 2x+1\,dx$

4 (b) $\int \frac{\cos x}{\sin^2 x}dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -1u^{-1} + C$

$u = \sin x$
$du = \cos x\,dx$

$= \boxed{-\frac{1}{\sin(x)} + C}$
$= \boxed{-\csc(x) + C}$

either answer is fine

3 (c) $\int \sec^2(x)\cdot e^{\tan(x)}dx = \int e^u\,du = e^u + C$

$u = \tan(x)$
$du = \sec^2(x)dx$

$= \boxed{e^{\tan(x)} + C}$

6 (d) $\int_0^1 \frac{x^4}{\sqrt[3]{x^5+1}}dx = \frac{1}{5}\int_1^2 \frac{1}{\sqrt[3]{u}}\,du = \frac{1}{5}\int_1^2 u^{-\frac{1}{3}}\,du$

$u = x^5 + 1$
$du = 5x^4\,dx$

$x=0 \quad u = 0^5 + 1 = 1$
$x=1 \quad u = 1^5 + 1 = 2$

$= \frac{1}{5}\frac{3}{2}u^{2/3}\Big|_1^2$

$= \frac{3}{10}\left[(2)^{2/3} - (1)^{2/3}\right] = \boxed{\frac{3}{10}\left[2^{\frac{2}{3}} - 1\right]}$

$\approx .176$

1. continued

6 (e) $\int_1^2 \frac{(\ln x)^3}{x}\,dx = \int_0^{\ln(2)} u^3\,du = \frac{1}{4}u^4 \Big|_0^{\ln(2)} = \frac{1}{4}\left[(\ln(2))^4 - (0)^4\right]$

$u = \ln(x)$

$du = \frac{1}{x}dx$

$x = 1 \Rightarrow u = \ln(1) = 0$

$x = 2 \Rightarrow u = \ln(2)$

$= \frac{(\ln(2))^4}{4}$

3. (10 pts) Consider the region between the curves $x = y^2 - 3$ and $y = \frac{1}{2}x$.

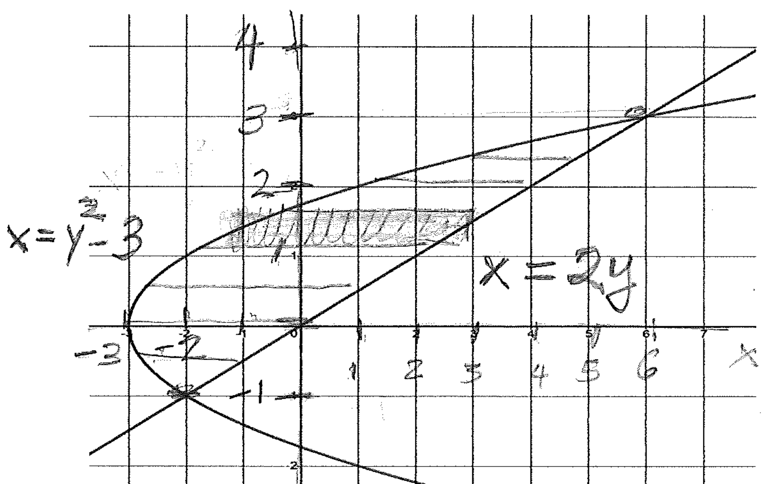
Set up and evaluate the integral(s) to find the area. You may use "by inspection" for the points of intersection.

Easiest: Slice by Δy

$A = \int_{-1}^{3} 2y - (y^2 - 3)\,dy$

$= \int_{-1}^{3} -y^2 + 2y + 3\,dy$

$= -\frac{1}{3}y^3 + y^2 + 3y \Big|_{-1}^{3}$



$= -\frac{1}{3}(3)^3 + (3)^2 + 3(3) - \left(-\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1)\right)$

$= 9 - \left(-\frac{5}{3}\right) = \frac{32}{3} = \boxed{10\tfrac{2}{3}\ \text{unit}^2}$

(Note: If you want to find the intersection points by hand (not by inspection) solve: $y^2 - 3 = 2y$

$y^2 - 2y - 3 = 0$

$(y-3)(y+1) = 0$

$y = 3$, $x = 2(3) = 6$; $y = -1$, $x = 2(-1) = -2$

Alternative set-up: Slice by Δx (WAY less efficient!)

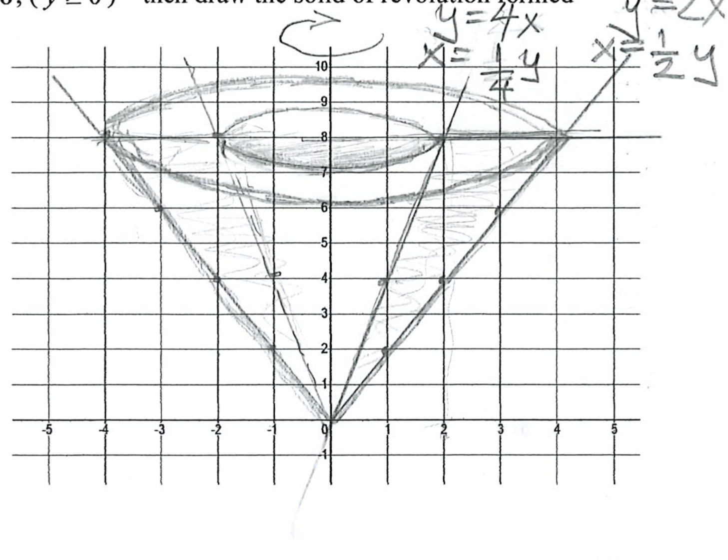
$A = \int_{-3}^{-2} \sqrt{x+3} - (-\sqrt{x+3})\,dx + \int_{-2}^{6} \sqrt{x+3} - \frac{1}{2}x\,dx$

$= \int_{-3}^{-2} 2(x+3)^{1/2}\,dx + \int_{-2}^{6} (x+3)^{1/2} - \frac{1}{2}x\,dx$

$= \frac{4}{3}(x+3)^{3/2} \Big|_{-3}^{-2} + \left[\frac{2}{3}(x+3)^{3/2} - \frac{1}{4}x^2\right]_{-2}^{6} = \text{a mess}$

$= 10\tfrac{2}{3}\ \text{unit}^2$

6. (24 pts) (a) Sketch the region bounded by $y=4x$, $y=2x$ and $y=8$, ($y \ge 0$) then draw the solid of revolution formed by revolving the region about the y-axis.



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$y=8 \Rightarrow x=2$ and $x=4$ are intersection points

$8=2x \Rightarrow x=4$

$8=4x \Rightarrow x=2$

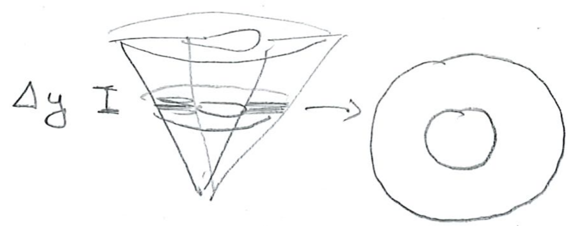
By inspection $y=2x$ and $y=4x$ intersect at $x=0$.

10 (b) Set up the integral or integrals needed to find the *volume* by each of the following methods.

the Disk/Washer Method Δy slices

$$\boxed{V=\pi\int_0^8 \left(\tfrac{1}{2}y\right)^2-\left(\tfrac{1}{4}y^2\right)dy}$$

$$=\pi\int_0^8 \frac{3}{16}y^2\,dy$$



$r_{inner}=x=\frac{1}{4}y$

$r_{outer}=x=\frac{1}{2}y$

10 the Shell Method

$$\boxed{V=2\pi\int_0^2 x(4x-2x)\,dx+\int_2^4 x(8-2x)\,dx}$$

$h=4x-2x$ $\quad h=8-2x$

$$=2\pi\int_0^2 2x^2\,dx+\int_2^4 8x-2x^2\,dx$$

7. (8 pts) Consider the curve $y=f(x)$, illustrated below. Explain/show how you can approximate the length, dL, of the **section** of the curve by using dx and dy. Fill in dL, dx, and dy on the graph.

$dL^2 \approx dx^2+dy^2$ by the Pythag. Theorem

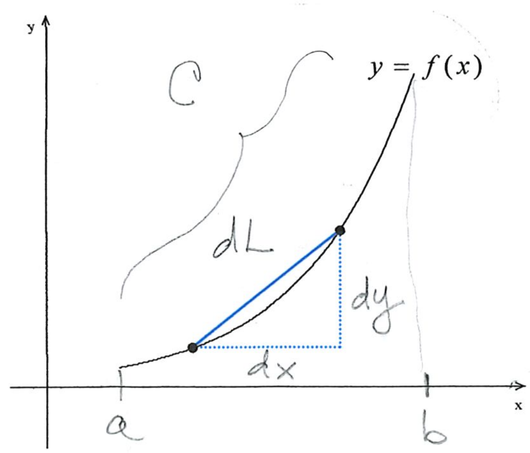
so $dL \approx \sqrt{dx^2+dy^2}$

Note: To get the full length of the curve, we have

$$L=\int_C dL=\int_a^b\sqrt{dx^2+dy^2}$$

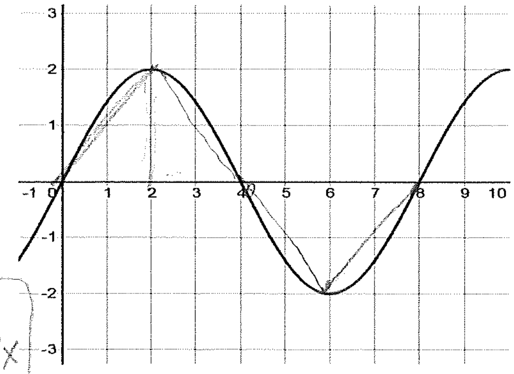
$$=\int_a^b\sqrt{\left(\frac{dx^2}{dx^2}+\frac{dy^2}{dx^2}\right)\cdot dx^2}$$

$$=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



8. (9 pts) Set up the integral to find the arclength of one period of the curve $y = 2\sin\left(\frac{\pi}{4}x\right) \quad 0 \le x \le 8$

Simplify the integrand, but you don't have to integrate.



$$y = 2\sin\left(\frac{\pi}{4}x\right)$$

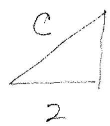
$$\frac{dy}{dx} = 2\cos\left(\frac{\pi}{4}x\right)\cdot\frac{\pi}{4}$$

$$= \frac{\pi}{4}\cos\left(\frac{\pi}{4}x\right)$$

$$L = \int_0^8 \sqrt{1+\left(\frac{\pi}{4}\cos\left(\frac{\pi}{4}x\right)\right)^2}\,dx$$

$$= \boxed{\int_0^8 \sqrt{1+\frac{\pi^2}{16}\cos^2\left(\frac{\pi}{4}x\right)}\,dx}$$

Extra credit (2 pts): Use triangles (illustrate) and geometry to estimate the curve length. (The length found using calculus is about 11.7 units, for comparison purposes).

$L \approx 4 \cdot$ hypotenuse of 

$c = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

$= 4(2\sqrt{2})$

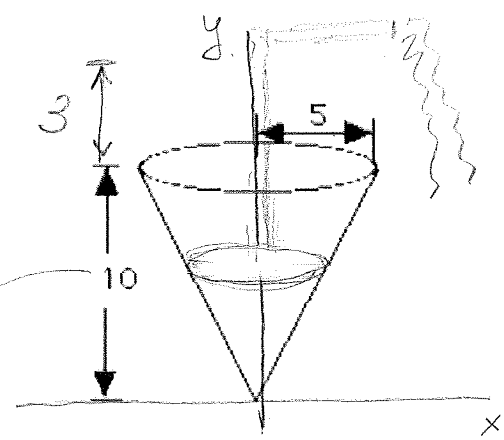
$= 8\sqrt{2} \approx 11.3$ — slight underestimate which makes sense geometrically

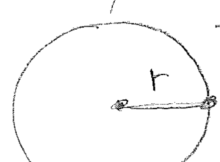
10. (12 pts) A tank is shaped like an inverted cone with height 10 meters and radius 5 meter (as shown below).

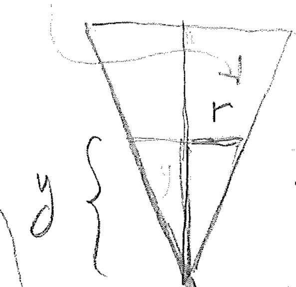
Set up *but don't evaluate* an integral to find the work done in pumping the water in a full tank to a point 3 meters above the tank. You may leave your answer in terms of ρ, π, and g

$$W = \int_0^{10} \rho g \pi \left(\tfrac{1}{2}y\right)^2 (13-y)\,dy$$

$$= \boxed{\frac{\rho g \pi}{4}\int_0^{10} 13y^2 - y^3\,dy}$$







$A = \pi r^2$

$A = \pi\left(\frac{1}{2}y\right)^2$

$\frac{r}{y} = \frac{5}{10}$

$r = \frac{1}{2}y$

Note: $\int_{y=\text{bottom slice}}^{y=\text{top slice}} \rho g \cdot A(y)\,D(y)\,dy \qquad \rho_{water} = 1000\,\frac{kg}{m^3} \qquad g \approx 9.8\,\frac{m}{s^2}$