Math 265B: Test 1 **Name:** KEY

For maximum credit, please show all of your work in a clear, logical manner. Simplify all answers as much as possible unless otherwise indicated. Only scientific calculators may be used on this exam.

1. (20 pts) Find each of the following:

(a) $\frac{1}{2}\int \frac{5x \cdot 2}{(x^2+1)^4}dx$

{let $u = x^2+1$
$du = 2x\ dx$}

$= \frac{5}{2}\int \frac{1}{u^4}du$

$= \frac{5}{2}\int u^{-4}du$

$= -\frac{1}{3}\frac{5}{2}u^{-3} + C$

$\boxed{= -\frac{5}{6}(x^2+1)^{-3} + C}$

$\boxed{\text{OR } -\frac{5}{6(x^2+1)^3} + C}$

(b) $\int \sin(3x)\cos^2(3x)dx$

{let $u = \cos(3x)$
$du = -3\sin(3x)dx$}

$= -\frac{1}{3}\int u^2 du$

$= -\frac{1}{3}\cdot\frac{1}{3}u^3 + C = \boxed{-\frac{1}{9}\cos^3(3x) + C}$

(c) $\int \frac{e^x}{1+e^x}dx$

{let $u = 1+e^x$
$du = e^x\ dx$}

$= \int \frac{1}{u}du$

$= \ln|u| + C$

$= \ln|1+e^x| + C$

note: Absolute value isn't necessary since $1+e^x$ is always positive

$= \boxed{\ln(1+e^x) + C}$

(d) $\int \frac{(\ln x)^2}{x}dx$

{let $u = \ln x$
$du = \frac{1}{x}dx$}

$= \int u^2 du.$

$= \frac{1}{3}u^3 + C$

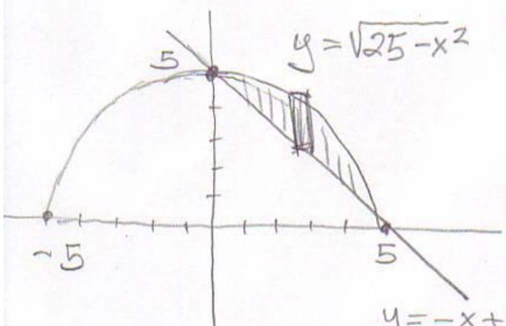
$\boxed{= \frac{1}{3}(\ln x)^3 + C}$

2. (8 pts) Evaluate the definite integral.

$$\int_0^1 xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{1}{2}e^{1^2} - \frac{1}{2}e^{0^2} = \boxed{\frac{1}{2}e - \frac{1}{2}}$$

OR use u-sub

3. (8 pts) Find the area between the curves $y=\sqrt{25-x^2}$ and $y=-x+5$.



$$\text{Area} = \int_0^5 \sqrt{25-x^2} - (-x+5)\,dx$$

$$= \int_0^5 \sqrt{25-x^2} + x - 5\,dx$$

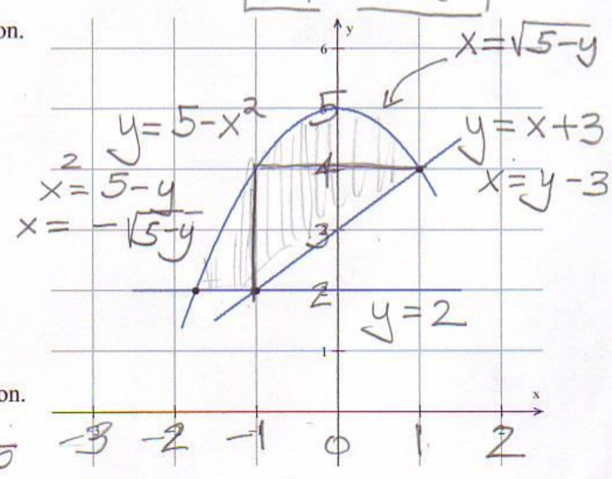
$$= \int_0^5 \sqrt{25-x^2}\,dx + \int_0^5 x - 5\,dx$$

using geometry

$$= \underline{\frac{1}{4}\pi(5)^2} + \frac{1}{2}x^2 - 5x\Big|_0^5$$

$$= \boxed{\frac{25\pi}{4} - \frac{25}{2}}$$

4. (12 pts) Consider the region bounded by $y=5-x^2$, $y=x+3$, and $y=2$, *as shown*.



(a) Set up the integral(s) with respect to x that give the *area* of this region.
You do not need to evaluate the integral(s).

$$\text{Area} = \int_{-2}^{-1} 5-x^2-2\,dx + \int_{-1}^{1} 5-x^2-(x+3)\,dx$$

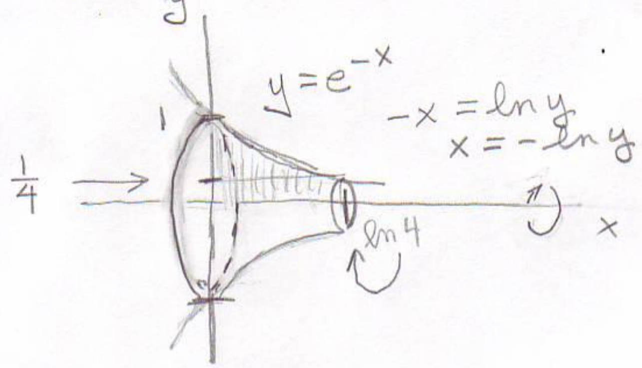
$$= \int_{-2}^{-1} 3-x^2\,dx + \int_{-1}^{1} 2-x-x^2\,dx$$

(b) Set up the integral(s) with respect to y that give the *area* of this region.
You do not need to evaluate the integral(s).

$$\text{Area} = \int_2^4 y-3-(-\sqrt{5-y})\,dy + \int_4^5 \sqrt{5-y}-(-\sqrt{5-y})\,dy$$

$$= \int_2^4 y-3+\sqrt{5-y}\,dy + \int_4^5 2\sqrt{5-y}\,dy$$

5. (18 pts) (a) Construct the solid of revolution formed by revolving the region bounded by $y = e^{-x}$, $y = 0$, $x = 0$ and $x = \ln 4$ about the x-axis.



(b) Set up the integral to find the *volume* by each of the following methods (you do not have to evaluate the integrals).

the Disk/Washer Method

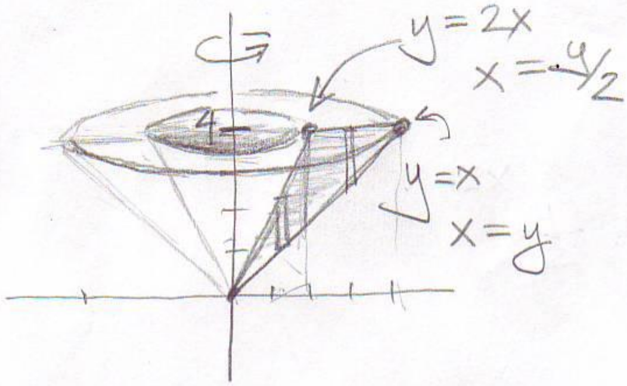
$$V = \int_0^{\ln 4} \pi (y)^2 \, dx = \int_0^{\ln 4} \pi (e^{-x})^2 \, dx = \int_0^{\ln 4} \pi e^{-2x} \, dx$$

the Shell Method

$$V = \int_0^{\frac{1}{4}} 2\pi \underbrace{\left(\tfrac{1}{4}\right)}_{\text{radius is fixed!}} \underbrace{\ln 4}_{\text{height is fixed!}} \, dy + \int_{\frac{1}{4}}^{1} 2\pi \underbrace{y}_{\text{radius of shell}} \underbrace{(-\ln y)}_{\text{height of shell}} \, dy$$

You could also simplify. use the volume formula for a cylinder.

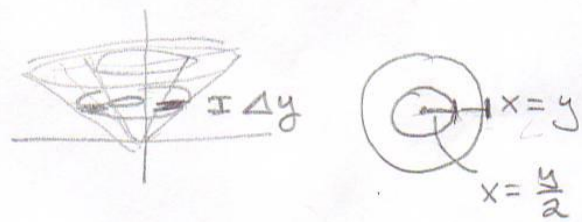
6. (18 pts) (a) Construct the solid of revolution formed by revolving the region bounded by $y = 2x$, $y = x$ and $y = 4$ about the y-axis.



(b) Set up the integral to find the ***volume*** by each of the following methods (you do not have to evaluate the integrals).

the Disk/Washer Method

$$V = \int_0^4 \pi(r_{outer})^2 - \pi(r_{inner})^2 \, dy$$



$$= \int_0^4 \pi(y)^2 - \pi\left(\frac{y}{2}\right)^2 dy$$

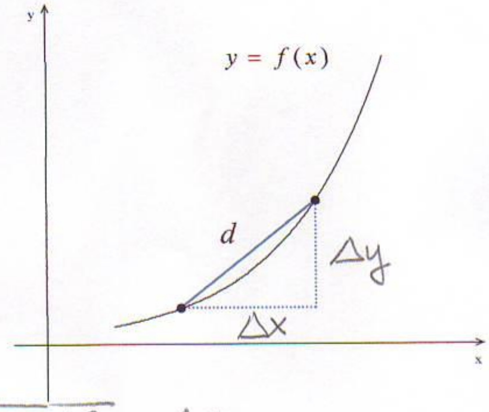
$$= \int_0^4 \pi y^2 - \frac{\pi y^2}{4} dy = \int_0^4 \frac{3\pi y^2}{4} dy$$

the Shell Method

$$V = \int_0^2 2\pi x\,(2x - x)\,dx + \int_2^4 2\pi x(4 - x)\,dx$$

$$= \int_0^2 2\pi x^2 \, dx + \int_2^4 8\pi x - 2\pi x^2 \, dx$$

7. (8 pts) Consider the curve $y = f(x)$, illustrated below. Explain/show how you can approximate the length, L, of the **piece** of the curve between the two points by relating Δx, Δy and d (as shown) with an appropriate equation then replacing Δy with a differential approximation. Simplify this expression as much as possible.



By the Pythagorean Theorem we have

$$(\Delta x)^2 + (\Delta y)^2 = d^2$$

But $\Delta y = f'(x)\Delta x$ $\not\approx 0$

$$d^2 = (\Delta x)^2 + (f'(x)\Delta x)^2$$

$$d = \sqrt{(\Delta x)^2 + (f'(x)\cdot\Delta x)^2}$$

$$= \sqrt{[1 + (f'(x))^2](\Delta x)^2} = \sqrt{1+(f'(x))^2}\cdot\Delta x$$

8. (10 pts) Set up the integral to find the arclength of the curve $y = \sqrt{1-x^2}$ for $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

Simplify the integrand as much as possible, but you don't have to integrate.

$$L = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \sqrt{1+(f'(x))^2}\,dx$$

$$y' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$(y')^2 = \frac{x^2}{1-x^2}$$

$$1+(y')^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \sqrt{\frac{1}{1-x^2}}\,dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}\,dx$$

Extra credit: (4 pts)

(a) Evaluate the integral you set up in # 8..

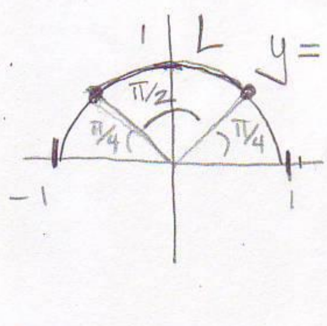
$$L = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}} = 2\int_{0}^{1/\sqrt{2}} \frac{1}{\sqrt{1-x^2}}\,dx = 2\sin^{-1}x \Big|_0^{1/\sqrt{2}}$$

even function

$$= 2\sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) - 2\sin^{-1}(0)$$

$$= 2\left(\tfrac{\pi}{4}\right) - 2\cdot(0) = \frac{\pi}{2}$$

(b) Verify the value you found in part (a) by finding the arclength using geometry.



$y = \sqrt{1-x^2}$ is a half-circle, hence arclength is the arclength formula for a circle. Radius is 1, angle subtended is $\frac{\pi}{2}$.

$$L = r\theta = 1\cdot\frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$