Math 265B: Test 1 **Name:** KEY

(60 points)

Credit is based on the correct work not just the final answer so please show your work in a clear, logical fashion. Simplify all answers as much as possible unless otherwise indicated and give EXACT answers unless an approximation is asked for in the problem. Only scientific calculators may be used on this exam.

1. (12 pts) Find each of the following:

(a) $\int \frac{t^2+5}{\sqrt{t}}\,dt$ $= \int \frac{t^2+5}{t^{1/2}}\,dt = \int t^{\frac{3}{2}} + 5t^{-\frac{1}{2}}\,dt$

$= \frac{2}{5}t^{\frac{5}{2}} + 5\cdot 2t^{\frac{1}{2}} + C$

$= \boxed{\frac{2}{5}t^{\frac{5}{2}} + 10t^{\frac{1}{2}} + C}$

(b) $\int \sec^2(x)e^{\tan(x)}dx$

Let $u = \tan(x)$

$du = \sec^2(x)\,dx$

$= \int e^u\,du$

$= e^u + C = \boxed{e^{\tan(x)} + C}$

Discuss

$\int \tan\theta \sec^2\theta\,d\theta$

$= \frac{1}{2}\tan^2\theta + C_1$

$\overset{?}{=} \frac{1}{2}\sec^2\theta + C_2$

(c) $-\int \frac{\sin(\theta)}{\cos^3(\theta)}\,d\theta$

$u = \cos\theta$, $du = -\sin\theta\,d\theta$

$= -\int \frac{1}{u^3}\,du = -\int u^{-3}\,du = -\left(-\frac{1}{2}u^{-2}\right) + C$

$= \frac{1}{2}(\cos\theta)^{-2} + C = \boxed{\frac{1}{2\cos^2\theta} + C}$

(d) $\frac{1}{2}\int \frac{2z+3}{z^2+6z}\,dz$

$u = z^2 + 6z$, $du = 2z + 6\,dz$

$= 2(z+3)\,dz$

$= \frac{1}{2}\int \frac{1}{u}\,du = \boxed{\frac{1}{2}\ln|z^2+6z| + C}$

$= \frac{1}{2}\ln|u| + C$

2. (5 pts) Evaluate the definite integral.

$$2\int_1^{\pi^2} \frac{\sin(\sqrt{x})}{2\sqrt{x}}\,dx$$

$u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{1}{2\sqrt{x}}\,dx$

$x = \pi^2$, $u = \sqrt{\pi^2} = \pi$

$x = 1$, $u = \sqrt{1} = 1$

You don't have to change the limits of integration

$= 2\int_1^{\pi} \sin(u)\,du$

$= 2\left[-\cos(u)\right|_1^{\pi}$

$= 2\left[-\cos(\pi) + \cos(1)\right] = \boxed{2 + 2\cos(1)}$

answer will be the same

3. (9 pts) (a) Find Trap(2) to estimate the value of. $\int_0^4 x^2 + 10\,dx$. Recall that the area of a trapezoid is $A = \frac{1}{2}(a+b)h$.

5½

Sketch a graph that illustrates Trap(2) for this integral.

$f(x) = x^2 + 10$

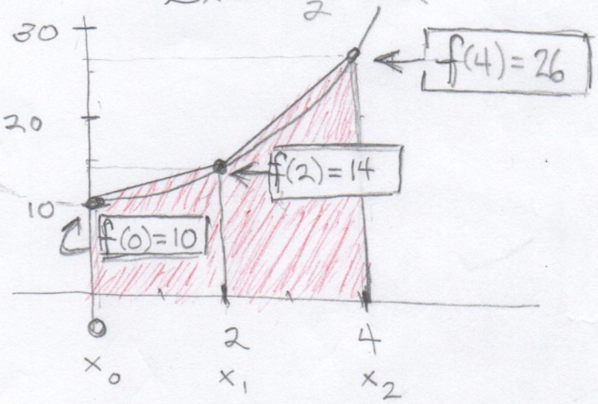
$\Delta x = \frac{4-0}{2} = 2$

$\text{Trap}(2) = \frac{1}{2}\left[f(x_0) + f(x_1)\right]\Delta x + \frac{1}{2}\left[f(x_1) + f(x_2)\right]\Delta x$

$= \left[\frac{1}{2}f(x_0) + f(x_1) + \frac{1}{2}f(x_2)\right]\Delta x$

$= \left[\frac{1}{2}f(0) + f(2) + \frac{1}{2}f(4)\right]\cdot 2$

$= \left[\frac{1}{2}(10) + 14 + \frac{1}{2}(26)\right]\cdot 2 = \boxed{64}$



3 (b) Find the error, absolute error, and percent error for the estimation in part (a):

Exact value:

$\int_0^4 x^2 + 10\,dx$

$= \frac{1}{3}x^3 + 10x\Big|_0^4 = \frac{64}{3} + 40 = \frac{184}{3} = 61\frac{1}{3}$

Error = Approx − Exact $= 64 - 61\frac{1}{3} = 2.\overline{6}$

Abs. Error = |Error| $= 2\frac{2}{3}$ (same!)

% Error $= \frac{|\text{Error}|}{|\text{Exact}|} = \frac{2\frac{2}{3}}{61\frac{1}{3}} \cong .044 = 4.4\%$

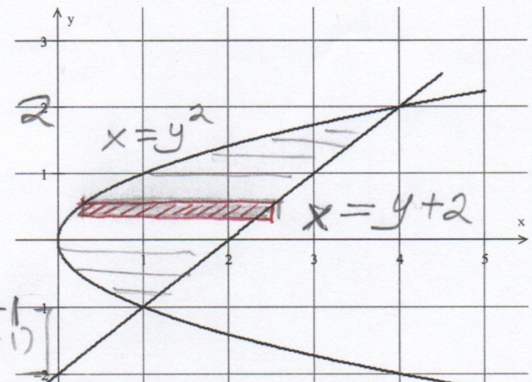
½ (c) If the number of subdivisions, n, were increased to be n = 20, by what factor would the error be reduced? Without attempting to find Trap(20), estimate the error for Trap(20).

$n = 2$ to $n = 20$ ⇒ divisions increased by factor of 10

Error(20) ≈ $.02\overline{6}$? so error decreases by factor of $10^2 = 100$

4. (6 pts) Find the area between the curves $x = y^2$ and $x = y + 2$ (see graph below for reference).

Slice by Δy (MUCH easier!)



$$A = \int_{-1}^{2} y+2-y^2\,dy$$

$$= -\tfrac{1}{3}y^3 + \tfrac{1}{2}y^2 + 2y \Big|_{-1}^{2}$$

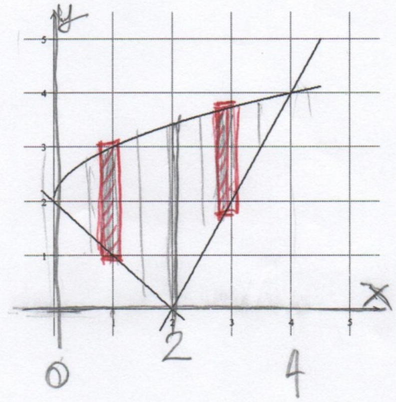
$$= -\tfrac{1}{3}(2)^3 + \tfrac{1}{2}(2)^2 + 2(2) - \left(-\tfrac{1}{3}(-1)^3 + \tfrac{1}{2}(-1)^2 + 2(-1)\right)$$

$$= -\tfrac{8}{3} + 6 - \tfrac{1}{3} - \tfrac{1}{2} + 2 = \boxed{4\tfrac{1}{2}}$$

reasonable based on "counting boxes"!

5. (6 pts) Consider the region bounded by $y = \sqrt{x} + 2$, $y = -x + 2$, and $y = 2x - 4$, as shown.

2 (a) Set up the integral(s) with respect to x that give the area of this region. Do simplify the integrand but you do not need to evaluate the integral(s).



$$A = \int_0^2 (\sqrt{x}+2) - (-x+2)\,dx + \int_2^4 (\sqrt{x}+2) - (2x-4)\,dx$$

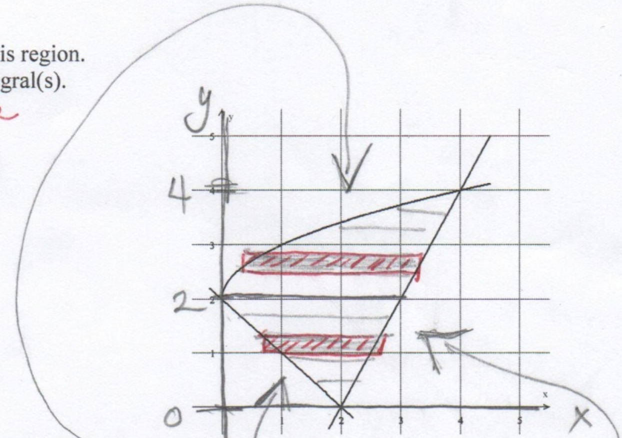
$$\boxed{A = \int_0^2 \sqrt{x} + x\,dx + \int_2^4 \sqrt{x} - 2x + 6\,dx}$$

4 (b) Set up the integral(s) with respect to y that give the *area* of this region. Do simplify the integrand but you do not need to evaluate the integral(s).

$$A = \int_0^2 \frac{y+4}{2} - (2-y)\,dy + \int_2^4 \frac{y+4}{2} - (y-2)^2\,dy$$

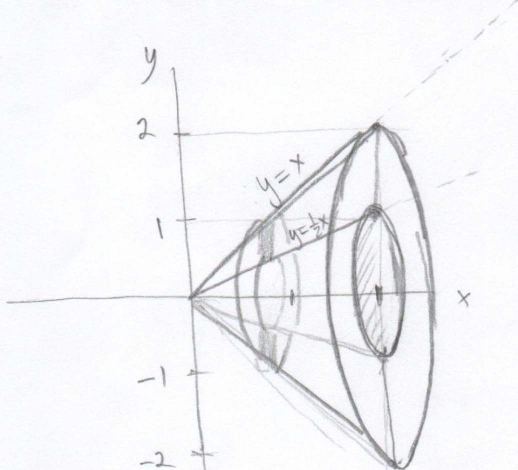
$-(y^2-4y+4)$ ok

$$= \int_0^2 \tfrac{3}{2}y\,dy + \int_2^4 -y^2 + \tfrac{9}{2}y - 2\,dy$$



$y=\sqrt{x}+2$	$y=-x+2$	$y=2x-4$
$x=(y-2)^2$	$x=2-y$	$x=\frac{y+4}{2}$

6. (6 pts) Use the Disk/Washer Method to find the volume of the solid formed by revolving the region bounded by $y=\frac{1}{2}x$, $y=x$, and $x=2$ about the x-axis. **Include a graph of the solid in your answer**!



$r_{outer} = x$

$r_{inner} = \frac{1}{2}x$

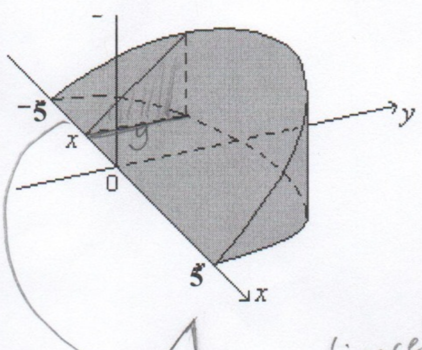
$$V=\int_0^2 \pi(x)^2-\pi\left(\tfrac{1}{2}x\right)^2dx$$

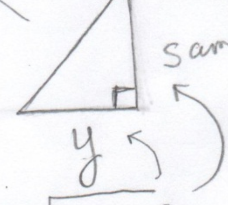
$$=\int_0^2 \pi\left(x^2-\tfrac{1}{4}x^2\right)dx$$

$$=\pi\int_0^2 \tfrac{3}{4}x^2\,dx$$

$$=\pi\left(\tfrac{3}{4}\cdot\tfrac{1}{3}x^3\right)\Big|_0^2 \quad \boxed{=2\pi}$$

7. (4 pts) A wooden wedge has a semi-circular base of radius r = 5cm. The cross section on any plane perpendicular to the diameter of the semi-circle is a right isosceles triangle with the right angle on the semi-circle (see illustration). Calculate the volume of the wedge. (Note: The equation of the base is $y=\sqrt{25-x^2}$.





same (isoceles)

y
$=\sqrt{25-x^2}$

$$V=\int_{-5}^{5} A(x)\,dx$$

$$=2\int_0^5 \left(\tfrac{1}{2}\left(\sqrt{25-x^2}\right)\left(\sqrt{25-x^2}\right)\right)dx$$

$$=\int_0^5 25-x^2\,dx$$

$$=25x-\tfrac{1}{3}x^3\Big|_0^5$$

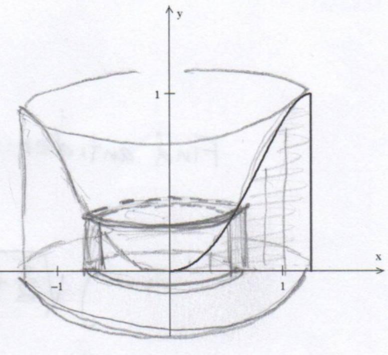
$$=25(5)-\tfrac{1}{3}(5)^3-0$$

$$=\frac{250}{3}$$

$$\boxed{=83\tfrac{1}{3}\text{ cm}^3}$$

$A(x)=\frac{1}{2}\text{base}\cdot\text{height}=\frac{1}{2}\left(\sqrt{25-x^2}\right)\left(\sqrt{25-x^2}\right)=\frac{1}{2}(25-x^2)$

8. (6 pts) Use the Cylindrical Shell Method to find the volume of the solid of revolution formed by revolving the region bounded by $y = \sin(x^2)$, $y = 0$ and $x = \sqrt{\frac{\pi}{2}}$ about the y-axis. (Region is shown in graph.)



Sketch the solid.

$$V = \int_0^{\sqrt{\pi/2}} 2\pi x \sin(x^2)\,dx$$

$u = x^2$
$du = 2x\,dx$
$x = 0 \Rightarrow u = 0^2 = 0$
$x = \sqrt{\frac{\pi}{2}} \Rightarrow u = \left(\sqrt{\frac{\pi}{2}}\right)^2 = \frac{\pi}{2}$

Cylinder radius $= x$
Cylinder height $= \sin(x^2)$

$$= \int_0^{\pi/2} \pi \sin(u)\,du$$

$$= \pi\left[-\cos(u)\Big|_0^{\pi/2}\right] = \pi\left[-\cos\left(\tfrac{\pi}{2}\right) - (-\cos(0))\right]$$

$$= \boxed{\pi}$$

9. (6 pts) Set up the integral to find the length of the curve $y = 2e^x - \frac{1}{8}e^{-x}$ for $0 \le x \le \ln 2$
Simplify the integrand as much as possible, but you don't have to integrate.

$$L = \int_0^{\ln 2} \sqrt{1 + [f'(x)]^2}\,dx$$

$y' = 2e^x + \frac{1}{8}e^{-x}$
$[y']^2 = \left(2e^x + \frac{1}{8}e^{-x}\right)^2$
$= 4e^{2x} + \frac{1}{2}e^0 + \frac{1}{64}e^{-2x}$

$$= \int_0^{\ln 2} \sqrt{1 + \left(4e^{2x} + \tfrac{1}{2} + \tfrac{1}{64}e^{-2x}\right)}\,dx$$

$$= \int_0^{\ln 2} \sqrt{\tfrac{3}{2} + 4e^{2x} + \tfrac{1}{64}e^{-2x}}\,dx$$

explain "simplify"

Extra credit (4 pts): Evaluate the integral you set up above.

over

1.) $\rho(x) = \begin{cases} 3x+1 & 0 \le x \le 1 \\ 4-x^2 & 1 \le x \le 2 \end{cases}$

$$\text{Mass} = \int_0^2 \rho(x)\,dx = \int_0^1 3x+1\,dx + \int_1^2 4-x^2\,dx$$

$$= \tfrac{3}{2}x^2 + x \Big|_0^1 + 4x - \tfrac{1}{3}x^3 \Big|_1^2$$

$$= \tfrac{3}{2}(1)^2 + (1) - 0 + \left[4(2) - \tfrac{8}{3}\right] - \left[4(1) - \tfrac{1}{3}\right]$$

$$= \tfrac{3}{2} + 1 + \tfrac{16}{3} - \tfrac{11}{3}$$

$$= \boxed{\tfrac{25}{6}\ g \text{ or } 4\tfrac{1}{6}\ g}$$

$$\text{Center of mass} = \frac{\sum \text{moments}}{\text{Total mass}}$$

where a "moment" is a mass-distance (it's the same as a weighted average)

$$\overline{x} = \frac{\int_0^2 x\rho(x)\,dx\ (*)}{\int_0^2 \rho(x)\,dx}$$

$$= \frac{15/4\ g\text{-cm}}{25/6\ g}$$

$$\boxed{= .9\ \text{cm}}$$

(*) Calculating the sum of ~~the~~ moments separately we have

$$\int_0^2 x\rho(x)\,dx$$

$$= \int_0^1 3x^2 + x\,dx + \int_1^2 4x - x^3\,dx$$

$$= x^3 + \tfrac{1}{2}x^2 \Big|_0^1 + 2x^2 - \tfrac{1}{4}x^4 \Big|_1^2$$

$$= 1 + \tfrac{1}{2} + 8 - 4 - (2 - \tfrac{1}{4}) = \tfrac{15}{4}\ g\text{-cm}$$

(a)

2. $F = kx$ and $F = 2.1 \times 10^6\ N$ for

$x = .2\ m$

so $\quad 2.1 \times 10^6 = k(.2)$

$\Longrightarrow k = 1.05 \times 10^7\ N/m$

(b) Energy (~~work~~) $= \int_0^{.5} F(x)\,dx$

$(\cancel{N}/\cancel{m})\ (m)\ (m) \Rightarrow$ final units N-m = Joules

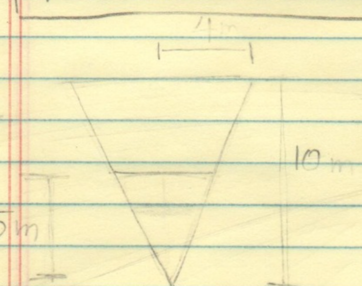
$= \int_0^{.5} 1.05(10)^7\, x\, dx$

$= 1.05(10)^7 \left[\frac{1}{2}x^2\right]\Big|_0^{.5}$

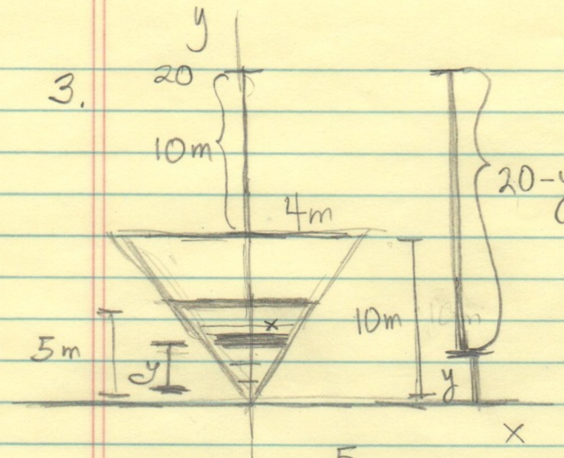
$= 1{,}312{,}500$ Joules

$\boxed{\text{The car absorbed } 1{,}312{,}500 \text{ J of energy}}$

3.

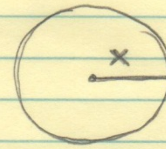


3.

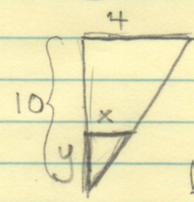


$\left.20-y\right\}$ = distance to outflow (m)

cross-section of "slice" of tank



$A=\pi x^2$ but what is x in terms of y?



$$\frac{x}{y}=\frac{4}{10}$$

by similer Δ's

$$\Longrightarrow \quad x=\frac{2}{5}y$$

so $A(y)=\pi\left(\frac{2}{5}y\right)^2$

$$=\frac{4\pi}{25}y^2$$

$$\text{Work} = \int_0^5 \rho g(20-y)A(y)\,dy$$

$$=\int_0^5 \rho g(20-y)\left(\frac{4\pi}{25}y^2\right)dy$$

$$=\frac{4\pi\rho g}{25}\int_0^5 20y^2-y^3\,dy$$

$$=\frac{4\pi\rho g}{25}\left[\frac{20}{3}y^3-\frac{1}{4}y^4\right|_0^5 \quad = \frac{4\pi\rho g}{25}\left[\frac{2500}{3}-\frac{625}{4}-0\right]$$

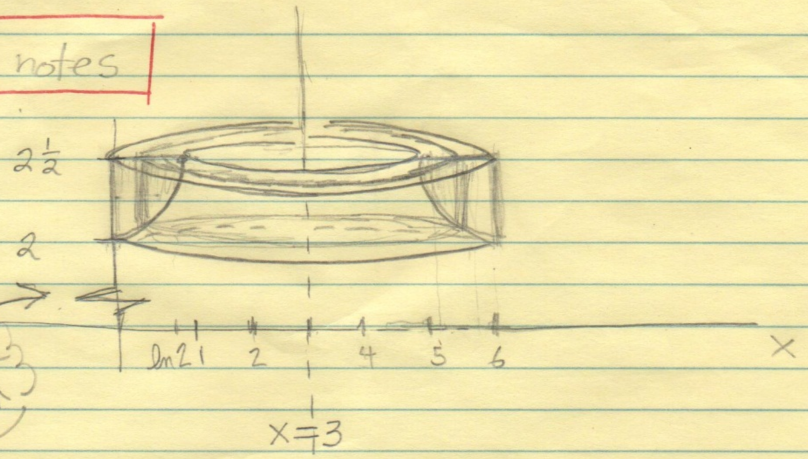
$$=\frac{325\rho g\pi}{3}\ \text{J}$$

Work done $=\dfrac{325\rho g\pi}{3}$ Joules

$\simeq$ 2,460,000 Joules

$\boxed{\text{4. See notes}}$

5.



Note: putting a break in the graph allows more control in scaling

Point of intersection:

Solve $\begin{cases} y = e^x + e^{-x} \\ y = \frac{5}{2} \end{cases} \Rightarrow \frac{5}{2} = e^x + e^{-x}$

$2e^x \left(\frac{5}{2} = e^x + \frac{1}{e^x} \right)$

$5e^x = 2e^{2x} + 2$

$0 = 2e^{2x} - 5e^x + 2$

$0 = (2e^x - 1)(e^x - 2)$

$e^x = \frac{1}{2}$ $\qquad$ $e^x = 2$

$x = \ln\left(\frac{1}{2}\right)$ $\qquad$ $\underline{\underline{x = \ln 2}}$

$(x \approx -.7)$

out of region

<u>Shells:</u>

$$V = \int_0^{\ln 2} 2\pi \overbrace{(3-x)}^{\text{radius of shell}} \overbrace{\left(\frac{5}{2} - \left(e^x + e^{-x}\right)\right)}^{\text{height of shell}} dx$$

$$= 2\pi \int_0^{\ln 2} (3-x)\left(\frac{5}{2} - e^x - e^{-x}\right) dx$$

using Wolfram to evaluate, we get

$$V \approx 4.0072 \text{ unit}^3$$