On 1(c), mention $\int_{-2}^{2} \sqrt{x^2}\, dx \neq 0$

Math 265B: Test 1 Name: KEY

Credit is based on the correct work not just the final answer so please show your work in a clear, logical fashion. Simplify all answers as much as possible unless otherwise indicated and give EXACT answers unless an approximation is asked for in the problem. Only scientific calculators may be used on this exam.

1. (15 pts) Find each of the following:

(a) $\int \frac{x^2}{(x^3+1)^4}dx = \frac{1}{3}\int \frac{1}{u^4}\,du = \frac{1}{3}\int u^{-4}\,du$

$u = x^3+1$

$du = 3x^2\,dx$

$= \frac{1}{3}\left(-\frac{1}{3}u^{-3}\right) + C$

$\boxed{= -\frac{1}{9(x^3+1)^3} + C}$

(b) $\int e^{2x}\cos(e^{2x})dx = \frac{1}{2}\int \cos(u)\,du$

$u = e^{2x}$

$du = 2e^{2x}\,dx$

$\boxed{= \frac{1}{2}\sin(e^{2x}) + C}$

(c) $\int \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}dx = I = \int \frac{\cos x}{\sqrt{\cos^2 x}}dx$

note: we're ignoring the fact that $\sqrt{\cos^2 x} = |\cos x|$

$= \int 1\,dx$

$\boxed{= x + C}$

2 approaches:

① Pythagorean Identity

$\cos^2 x + \sin^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

② Let $u = \sin x$

$du = \cos x$

$I = \int \frac{1}{\sqrt{1-u^2}}\,du$

$= \sin^{-1}(u) + C$

$= \sin^{-1}(\sin x) + C$

$\boxed{= x + C}$

2. (8 pts) Evaluate the definite integral.

$\int_1^5 \frac{\ln(x)}{x}dx = \int_0^{\ln(5)} u\,du$

$= \frac{1}{2}u^2 \Big|_0^{\ln(5)}$

$u = \ln x$

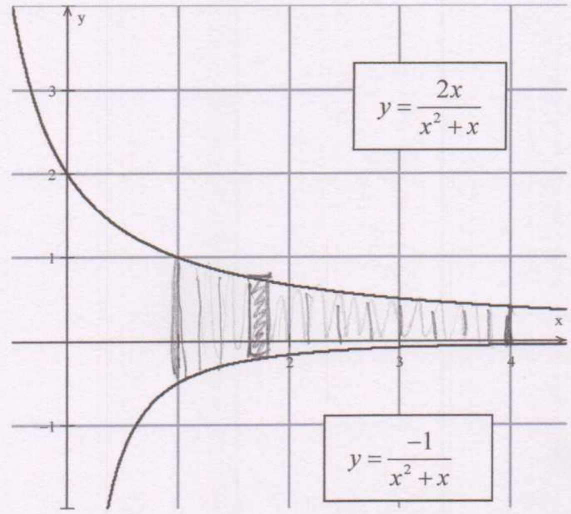
$du = \frac{1}{x}$

$x = 1 \Rightarrow u = \ln(1) = 0$

$x = 5 \Rightarrow u = \ln(5)$

$\boxed{= \frac{1}{2}\left[\ln(5)\right]^2}$

3. (6 pts) Use calculus to find the exact area between the curves
$y=\frac{2x}{x^2+x}$ and $y=\frac{-1}{x^2+x}$ from x = 1 to x = 4 (see graph for reference).



$$\text{Area} = \int_1^4 \frac{2x}{x^2+x} - \left(\frac{-1}{x^2+x}\right) dx$$

$$= \int_1^4 \frac{\overbrace{2x+1}^{du}}{\underbrace{x^2+x}_{u}}\, dx$$

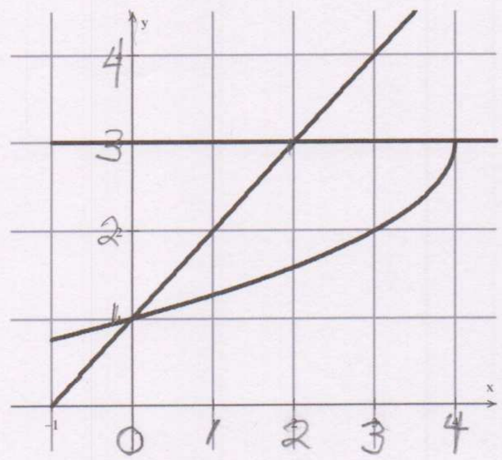
$$= \ln|x^2+x| \Big|_1^4$$

$$= \ln(20) - \ln(2) = \boxed{\ln(10)}$$

Best answer

4. (10 pts) Consider the region bounded by $y=3-\sqrt{4-x}$, $y=x+1$, and $y=3$, as shown.

(a) Set up but don't evaluate the integral(s) **with respect to x** that give the area of this region.



$$\text{Area} = \int_0^2 (x+1) - (3-\sqrt{4-x})\, dx + \int_2^4 3 - (3-\sqrt{4-x})\, dx$$

$$= \int_0^2 x - 2 - \sqrt{4-x}\, dx + \int_2^4 \sqrt{4-x}\, dx$$

(b) Set up but don't evaluate the integral(s) **with respect to y** that give the *area* of this region. Do simplify the integrand but you do not need to evaluate the integral(s).

$$\text{Area} = \int_1^3 4 - (y-3)^2 - (y-1)\, dy = \int_1^3 5 - (y-3)^2 - y\, dy$$

$y = 3 - \sqrt{4-x}$

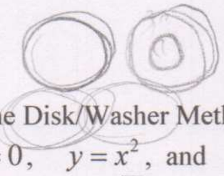
$y - 3 = -\sqrt{4-x}$

$(y-3)^2 = 4 - x$

$x = 4 - (y-3)^2$

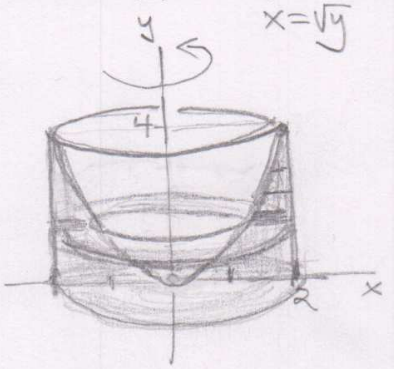
$y = x + 1$

$x = y - 1$



5. (9 pts) Use the Disk/Washer Method to find the volume of the solid formed by revolving the region bounded by $y=0$, $y=x^2$, and $x=2$ about the y-axis. **Include a sketch of the solid in your answer**!

$x=\sqrt{y}$



$$V=\int_{y=0}^{y=4}\pi(2)^2-\pi(\sqrt{y})^2\,dy$$

$$=\pi\int_0^4 4-y\,dy$$

$$=\pi\left[4y-\tfrac{1}{2}y^2\right|_0^4$$

$$=\boxed{8\pi \text{ unit}^3}$$

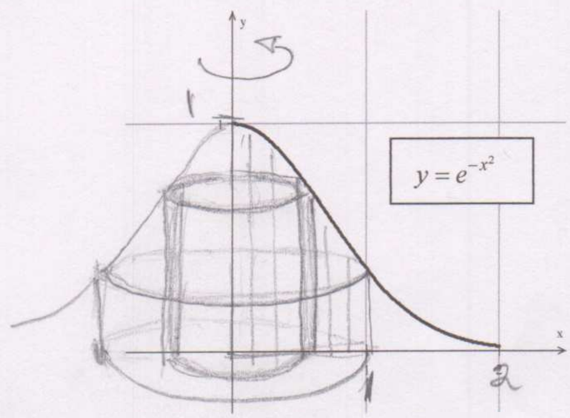
6. (9 pts) Use the Cylindrical Shell Method to find the volume of the solid of revolution formed by revolving the region bounded by $y=e^{-x^2}$ and $y=0$ from $x=0$ to 1 about the y-axis. **Include a sketch of the solid in your answer**!

$$V=\int_0^1 2\pi x e^{-x^2}\,dx$$

$$=-\pi e^{-x^2}\Big|_0^1$$

$$=-\pi e^{-1}-(-\pi e^0)$$

$$=\boxed{\pi(1-e^{-1})} \text{ unit}^3$$



7. (6 pts) Consider the curve $y=f(x)$, illustrated below. Explain/show how you can approximate the length, d, of the **piece** of the curve between the two points by relating Δx, Δy and d (as shown) with an appropriate equation then replacing Δy with a linear approximation. Simplify this expression as much as possible.

Linear approximation: slope of secant $\approx$ slope of tangent

$$\frac{\Delta y}{\Delta x}\approx\frac{dy}{dx}=f'(x)$$

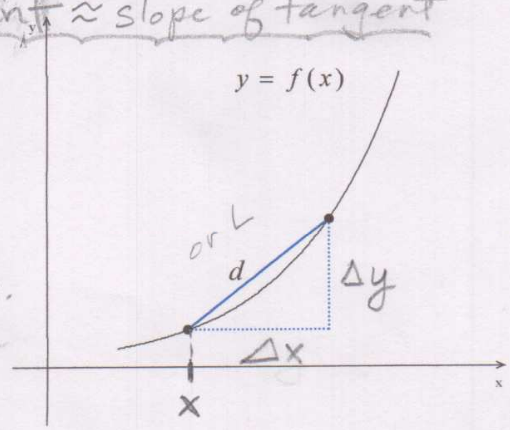
so $\Delta y\approx f'(x)\Delta x$

$$d=\sqrt{(\Delta x)^2+(\Delta y)^2} \quad \text{by the Pythag. Theorem}$$

(or L)

$$=\sqrt{(\Delta x)^2+(f'(x)\Delta x)^2}$$

$$\boxed{d=\sqrt{1+(f'(x))^2}\cdot\Delta x}$$



8. (6 pts) A thin rod that is 4 cm long has density (in grams per centimeter) that varies along its length according to the function $\rho(x)=\begin{cases} x+2 \ {}^{g}/_{cm} & 0\le x\le 2 \ cm \\ x^2 \ {}^{g}/_{cm} & 2\le x\le 4 \ cm \end{cases}$

Set up *but don't evaluate* integral(s) that give the total mass of the rod.

$$\boxed{\text{Mass} = \int_0^2 x+2\,dx + \int_2^4 x^2\,dx}$$

9. (10 pts) Cars are designed with a "crumple zone" in the front of the car. The more "crushed" the front becomes in the event of a crash, the more it will resist further crushing, so we can model the forces and energy involved in the same way we do a linear spring. In a safety test, it was found that a for certain type of car, a force of 1.4×10^6 Newtons would compress the front of the car 0.1 meters.

(a) Assuming the crumple zone obeys Hooke's Law, find the spring constant, k, for this type of car.

$F = kx$

$F = 1.4\times10^6\,N$

$x = .1\,m$

$1.4(10)^6\,N = k(.1m)$

$$\boxed{k = 1.4(10)^7\,\frac{N}{m}}$$

(b) If the same type of car is in an accident where the front of the car crumples (compresses) 0.3 meter, use calculus to determine the amount of energy the crumple zone absorbed.

$$W = \int_0^{.3} 1.4(10)^7\,x\,dx$$

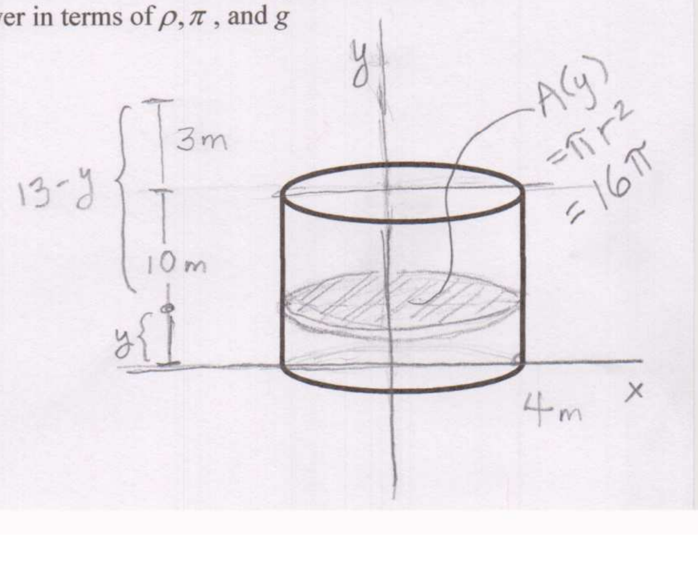
$$= 1.4(10)^7\left(\tfrac{1}{2}x^2\right)\Big|_0^{.3}$$

$$= 1.4(10)^7\,\tfrac{1}{2}(.3)^2 = \boxed{6.3\times10^5\,J}$$

10. (8 pts) Set up *but don't evaluate* an integral to find the work done in pumping the water in a full tank in the shape of a cylinder, resting on its base (*as shown),* with radius of 4 meters and height of 10 meters to a point 3 meters above the tank. You may leave your answer in terms of ρ,π, and g

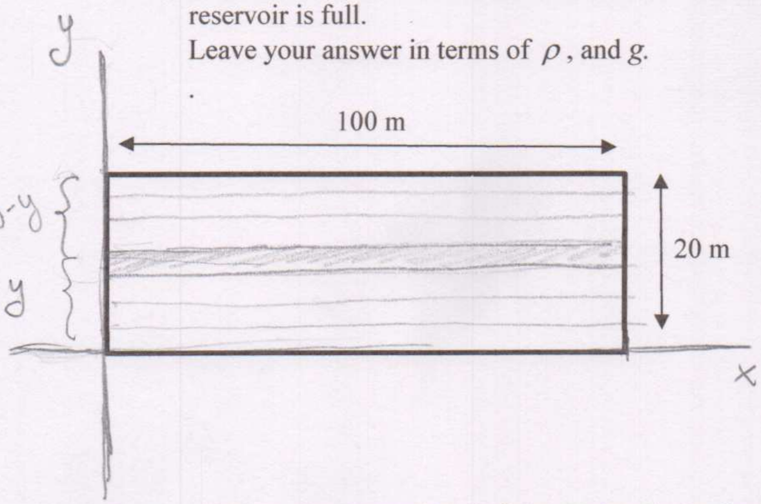
$$W = \int_0^{10} \rho g(16\cdot\pi)(13-y)\,dy$$

$$\boxed{= \int_0^{10} 16\rho g\pi(13-y)\,dy}$$



11.(8 pts) Set up *but don't evaluate* an integral that gives the force that the water in a reservoir exerts on a rectangular dam with the following dimensions: height is 20 meters, length is 100 meters. Assume the reservoir is full.

Leave your answer in terms of ρ, and g.



$$\text{Force} = \int_0^{20} \rho g (20-y)\, 100\, dy$$

$$\boxed{= \int_0^{20} 100 \rho g\, (20-y)\, dy}$$

12. (5 pts) Set up the integral to find the arc length for the curve $y = x^2 - \frac{1}{8}\ln(x)$ from x = 1 to x = 2

Simplify the integrand, but you don't have to evaluate the integral.

$$y' = 2x - \frac{1}{8x}$$

$$[y']^2 = \left(2x - \frac{1}{8x}\right)^2$$

$$= 4x^2 - 2(2x)\left(\frac{1}{8x}\right) + \frac{1}{64x^2}$$

$$= 4x^2 - \frac{1}{2} + \frac{1}{64x^2}$$

$$L = \int_1^2 \sqrt{\underbrace{1 + [f'(x)]^2}_{(*)}}\, dx$$

$$\boxed{= \int_1^2 \sqrt{4x^2 + \frac{1}{2} + \frac{1}{64x^2}}\, dx}$$

$$(*)\ 1 + [y']^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64}x^2 = 4x^2 + \frac{1}{2} + \frac{1}{64x^2}$$

Extra credit (3 pts): Evaluate the integral you set up above.

$$L = \int_1^2 \sqrt{4x^2 + \frac{1}{2} + \frac{1}{64x^2}}\, dx$$

$$= \int_1^2 \sqrt{\left(2x + \frac{1}{8x}\right)^2}\, dx$$

$$= \int_1^2 2x + \frac{1}{8x}\, dx$$

$$\rightarrow x^2 + \frac{1}{8}\ln|x| \Big|_1^2$$

$$= (2)^2 + \frac{1}{8}\ln(2) - \left(1^2 + \cancel{\frac{1}{8}\ln(1)}^{\,0}\right)$$

$$\boxed{= 3 + \frac{1}{8}\ln(2)}$$