Math 265B: Test 2 **Name:** KEY

For maximum credit, please show all of your work in a clear, logical manner. Simplify all answers as much as possible. (70 points)-

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Formulas: $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$ $\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

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1. (8 pts) Given the integral $\int \frac{x}{\sqrt{25-x^2}}\,dx$

(a) Could you use Partial Fraction Decomposition to integrate? YES NO Why or why not? denominator not a polynomial (the $\sqrt{\ }$ ruins it)

(b) Could you use Trig Substitution? YES NO If yes, what would the substitution be?

$x = 5\sin\theta$

(c) Could you use u-Substitution (or Reverse the Chain Rule/Guess-and-Check)? YES NO

(d) Could you manipulate the integrand to be a simpler function? YES NO !!!
If yes, show what you would do:

Do the integration using the most efficient method:

Let $u = 25 - x^2$
$du = -2x\,dx$

$-\frac{1}{2}\int \frac{-2x}{\sqrt{25-x^2}}\,dx$

$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= -\frac{1}{2}\cdot\frac{2}{1}u^{\frac{1}{2}} + C$

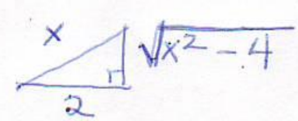
$= -u^{\frac{1}{2}} + C$

$= \boxed{-\sqrt{25-x^2} + C}$

2. (2 pts) Suppose you made the Trig Substitution of $x = 2\sec\theta$ in an integral.

Which of the following would be true? Show work for credit! $\sec\theta = \frac{x}{2}$

a) $\cos\theta = \frac{2}{x}$ b) $\tan\theta = \frac{\sqrt{x^2+4}}{2}$ no



c) both of these d) neither of these

3. (5 pts) Use Integration by Parts (the formula, not the Column Method) to integrate:

$\int x^3 \ln(x)\, dx$.

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{4}x^4$

$$\int u\,dv = u \cdot v - \int v\,du$$

$$\int x^3 \ln(x) dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \left(\frac{1}{x}\, dx\right)$$

$$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 dx$$

$$= \boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$$

4. (7 pts) Use Partial Fraction Decomposition to integrate:

$\int \frac{5x^3+4x+2}{x^2(x^2+1)} dx$.

$$\frac{5x^3+4x+2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$5x^3 + 4x + 2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$5x^3 = Ax^3 + Cx^3$

$0x^2 = Bx^2 + Dx^2$

$4x = Ax$

$2 = B$

$A + C = 5$

$B + D = 0$

$A = 4$

$C = 1$

$D = -2$

$$= \int \frac{4}{x} + \frac{2}{x^2} + \frac{x-2}{x^2+1}\, dx$$

$$= \int \frac{4}{x} + 2x^{-2} + \frac{x}{x^2+1} - \frac{2}{x^2+1}\, dx$$

$$= 4\ln|x| - 2x^{-1} + \frac{1}{2}\ln|x^2+1| - 2\tan^{-1}(x) + C$$

OR

$$= -\frac{2}{x} - 2\tan^{-1}(x) + \ln\left(x^4\sqrt{x^2+1}\right) + C$$

absoute value is not needed...

5. (6 pts) Use Identities to integrate:

$\int \sin^3(\phi)\cos^2(\phi)\, d\phi$.

$u = \cos\phi$

$du = -\sin\phi\, d\phi$

$= \int \sin^2\phi \cos^2\phi \sin\phi \, d\phi$

$= \int (1-\cos^2\phi)\cos^2\phi \sin\phi\, d\phi$

$= -\int (1-u^2)u^2\, du$

$= \int u^4 - u^2 \, du$

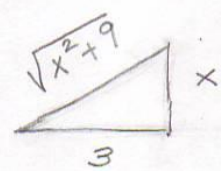
$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \boxed{\frac{1}{5}\cos^5\phi - \frac{1}{3}\cos^3\phi + C}$

6. (8 pts) Use Trig Substitution to evaluate the definite integral: $\int_0^4 \frac{1}{(x^2+9)^{\frac{3}{2}}}\, dx$.

Set $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$



$= \int_{x=0}^{x=4} \frac{1}{(9\tan^2\theta + 9)^{3/2}}(3\sec^2\theta\, d\theta)$

$= \int_{x=0}^{x=4} \frac{3\sec^2\theta \;\; d\theta}{(9(\tan^2+1))^{3/2}}$

$= \int_{x=0}^{x=4} \frac{3\sec^2\theta}{27\sec^3\theta}\, d\theta$

$= \frac{1}{9}\int_{x=0}^{x=4} \cos\theta\, d\theta$

$= \frac{1}{9}\sin\theta \Big|_{x=0}^{x=4}$

$= \frac{1}{9}\frac{x}{\sqrt{x^2+9}} \Big|_0^4$

$= \frac{1}{9}\left[\frac{4}{5} - \frac{0}{3}\right]$

$= \boxed{\frac{4}{45}}$

even power $\Rightarrow \frac{1}{2}$ angle ID

7. (4 pts) Use Identities to integrate: $\int \sin^2 \omega t \ \ dt$, where ω is a real-valued constant.

$$= \frac{1}{2}\int 1 - \cos 2\omega t \ dt$$

$$= \frac{1}{2}\left[t - \frac{1}{2\omega}\sin 2\omega t\right] + C$$

$$\boxed{= \frac{1}{2}t - \frac{1}{4\omega}\sin 2\omega t + C}$$

8. (2 pts) How would you check that the following integration was done correctly? Differentiate!
Perform the check and write a conclusion:

$$\int \frac{x^2+1}{x^3+x}\,dx = \frac{1}{3}\ln(x^3+x)+C$$

$$\frac{d}{dx}\left[\frac{1}{3}\ln(x^3+x)\right]$$

$$= \frac{1}{3}\left(\frac{1}{x^3+x}\right)\cdot 3x^2+1 \neq \frac{x^2+1}{x^3+x}$$

Conclusion: not the correct antiderivative.

9. (10 pts) Determine which of the integration techniques listed below would be the ***BEST*** to apply to find the following integrals and how you would begin to employ that technique. You do not have to integrate.

Techniques:
Partial Fraction Decomposition (write what the decomposition would be)

Integration by Parts (write your choice for u and dv)

u-Substitution or Reverse the Chain Rule (write either u or what the form the antiderivative would be in)

Trig Substitution (write what the substitution would be)

Trig Identities (write what identity/identities you would use…see last page)

a.) $\int x^3 \cos(x)\,dx$ Best Technique/next step: Integration by Parts: $u = x^3$, $dv = \cos(x)\,dx$

b.) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ Best Technique/next step: U-Sub / Reverse Chain: $u = \sqrt{x}$ OR guess $\sin\sqrt{x}$

c.) $\int \frac{1}{(x-3)(x+1)}\,dx$ Best Technique/next step: PFD: $\frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

d.) $\int \cos^4(x)\sin^2(x)dx$ Best Technique/next step: Trig ID's ($\frac{1}{2}$angle): $\cos^2 x = 1/2(1+\cos 2x)$, $\sin^2 x = 1/2(1-\cos 2x)$

e.) $\int \frac{1}{\sqrt{x^2-4}}\,dx$ Best Technique/next step: Trig Sub: $x = 2\sec\theta$

10. (6 pts) Evaluate the definite integral. Leave your answer exact (in terms of "e").

$$\int_0^1 x^3 e^{x^2}\,dx = \frac{1}{2}x^2 e^{x^2} - \int_0^1 \cancel{2}x\left(\frac{1}{\cancel{2}}e^{x^2}\right)dx$$

u	dv
x^2	xe^{x^2}
$2x$	$\frac{1}{2}e^{x^2}$

$$= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2}\Big|_0^1$$

$$= \cancel{\frac{1}{2}(1)^2 e^{(1)^2} - \frac{1}{2}e^{(1)}} - \left[\frac{1}{2}(0)^2 e^{0^2} - \frac{1}{2}e^{0^2}\right]$$

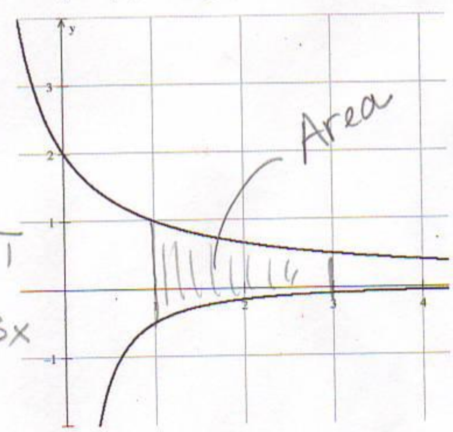
$$= \frac{1}{2}(1)$$

$$\boxed{= \frac{1}{2}}$$

11. (6 pts) Use calculus to find the area between $f(x) = \frac{2x}{x^2+x}$ and $g(x) = \frac{-1}{x^2+x}$ on the interval $1 \le x \le 3$.

Give the exact value, then an approximation to 2 decimal places **Hint**: $x^2 + x = x(x+1)$

Graph of f and g (for reference)



$$\text{Area} = \int_1^3 \frac{2x}{x^2+x} - \frac{-1}{x^2+x}\,dx$$

$$= \int_1^3 \frac{2x+1}{x(x+1)}\,dx$$

PFD!

$$\frac{2x+1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$2x+1 = A(x+1) + Bx$$

$$2 = A + B$$

$$1 = A$$

$$B = 1$$

$$= \int_1^3 \frac{1}{x} + \frac{1}{x+1}\,dx$$

$$= \ln|x| + \ln|x+1| \Big|_1^3$$

$$= \ln(3) + \ln(4) - (\cancel{\ln(1)}^{\,0} + \ln(2))$$

$$= \ln\left(\frac{12}{2}\right) = \ln 6 \approx 1.79$$

Note: $\int \frac{2x+1}{x^2+x}\,dx$ is a $\int \frac{du}{u}$ integral! I didn't notice that...

Extra credit (4 points)

Integrate, using any (or all!) methods:

$$\int \frac{e^x}{e^{3x}\sqrt{e^{2x}-4}}\,dx$$

let $u = e^x$

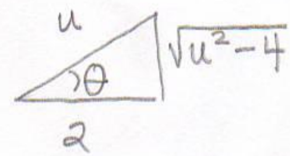
$du = e^x\,dx$

$$= \int \frac{1}{u^3\sqrt{u^2-4}}\,du$$

Let $u = 2\sec\theta$

$du = 2\sec\theta\tan\theta\,d\theta$

$\frac{u}{2} = \sec\theta$



$\theta = \sec^{-1}\left(\frac{u}{2}\right)$

$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{8\sec^3\theta\sqrt{4\sec^2\theta-4}}$$

$$= \int \frac{\cancel{\sec}\theta\tan\theta\,d\theta}{4\sec^{\cancel{3}}\theta\cdot\sqrt{4(\sec^2\theta-1)}}$$

$$= \frac{1}{8}\int \frac{\cancel{\tan\theta}}{\sec^2\theta\cdot\cancel{\tan\theta}}\,d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{16}\int 1+\cos 2\theta\,d\theta$$

$$= \frac{1}{16}\theta + \frac{1}{32}\sin 2\theta + C$$

$$= \frac{1}{16}\theta + \frac{1}{16}\sin\theta\cos\theta + C$$

$$= \frac{1}{16}\sec^{-1}\left(\frac{u}{2}\right) + \frac{1}{16}\left(\frac{\sqrt{u^2-4}}{u}\right)\left(\frac{2}{u}\right) + C$$

$$\boxed{= \frac{1}{16}\sec^{-1}\left(\frac{e^x}{2}\right) + \frac{1}{8}\,\frac{\sqrt{e^{2x}-4}}{e^{2x}} + C}$$