3(b) make bottom limit 0
add formula

Math 265B: Test 2 **Name:** KEY
(100 points)

Credit for each problem is based on the amount of correct work shown, not just the final answer Simplify all answers as much as possible but for full credit, leave answers in exact terms. Only scientific calculators may be used on this exam.

Formulas: $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$ $\quad \cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\quad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

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1. (10 pts) Integrate. Suggested method: Integration by Parts.

$\int 5x^7 \ln(x^3)\,dx$.

$u = \ln(x^3)$ | $dv = 5x^7\,dx$

$du = \frac{1}{x^3}\cdot 3x^2\,dx$ | $v = \frac{5}{8}x^8$

$du = \frac{3}{x}\,dx$

$= \frac{5}{8}x^8\ln(x^3) - \int \frac{5}{8}x^8\left(\frac{3}{x}\,dx\right)$

$= \frac{5}{8}x^8\ln(x^3) - \int \frac{15}{8}x^7\,dx$

$\boxed{= \frac{5}{8}x^8\ln(x^3) - \frac{15}{64}x^8 + C}$

2. (10 pts) Integrate. Suggested Method: Pythagorean Trig Identity.

$\int \cos^3(\theta)\sin^2(\theta)\,d\theta$

odd $\Rightarrow$ peel off a $\cos\theta$

Let $u = \sin\theta$
$du = \cos\theta\,d\theta$

$= \int \cos^2\theta\,\sin^2\theta\cdot\cos\theta\,d\theta$

$= \int (1-\sin^2\theta)\sin^2\theta\,\cos\theta\,d\theta$

$= \int (1-u^2)u^2\,du$

$= \int u^2 - u^4\,du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \boxed{= \frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta + C}$

3(b) make bottom limit $\underline{\underline{0}}$

3. (18 pts) Integrate: Suggested method: Trig Substitution

10 (a) $\int \sqrt{16-x^2}\,dx$ (*)

$x = 4\sin\theta \qquad x^2 = 16\sin^2\theta$

$dx = 4\cos\theta\, d\theta \qquad \sqrt{16-16\sin^2\theta}$ (*)

$= \sqrt{16(1-\sin^2\theta)}$

$= 4\sqrt{\cos^2\theta}$

$= 4\cos\theta$

$$= \int 4\cos\theta \cdot 4\cos\theta\, d\theta$$

$$= 16\int \cos^2\theta\, d\theta$$

5 pts

$$= 16\int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 8\left[\theta + \tfrac{1}{2}\sin 2\theta\right] + C$$

2 pts

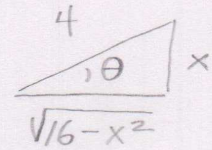
$$= 8\theta + 4(2\sin\theta\cos\theta) + C$$

3 pts

$$= 8\sin^{-1}\left(\tfrac{x}{4}\right) + 8\left(\tfrac{x}{4}\right)\left(\tfrac{\sqrt{16-x^2}}{4}\right) + C$$

$$\boxed{= 8\sin^{-1}\left(\tfrac{x}{4}\right) + \tfrac{1}{2}x\sqrt{16-x^2} + C}$$

Triangle:



$x = 4\sin\theta$

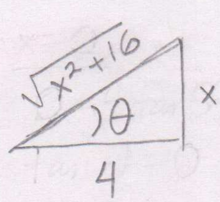
$\frac{x}{4} = \sin\theta$

$\theta = \sin^{-1}\left(\frac{x}{4}\right)$

8 (b) $\int_{0}^{4} \frac{1}{(x^2+16)^{3/2}}\,dx$ (*)

$x = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$



(*) $(16\tan^2\theta + 16)^{3/2}$

$= (16\cdot(\tan^2\theta + 1))^{3/2}$

$= (16\sec^2\theta)^{3/2}$

$= (4\sec\theta)^3$

$= 64\sec^3\theta$

$$= \int_{x=0}^{x=4} \frac{4\sec^2\theta\, d\theta}{64\sec^3\theta}$$

$$= \frac{1}{16}\int_{x=0}^{x=4} \frac{1}{\sec\theta}\, d\theta$$

4

$$= \frac{1}{16}\int_{x=0}^{x=4} \cos\theta\, d\theta$$

4

$$= \frac{1}{16}\sin\theta \Big|_{x=0}^{x=4} = \frac{x}{16\sqrt{x^2+16}}\Bigg|_0^4 = \frac{1}{\cancel{16}_{4}}\frac{\cancel{4}^{1}}{\sqrt{32}} - 0 = \frac{1}{16\sqrt{2}} \text{ or } \frac{\sqrt{2}}{32}$$

4. (16 pts) Integrate. Suggested method: Partial Fraction Decomposition and/or Long Division.

10 (a) $\int \frac{4x+2}{x^3+9x}dx$. $= \int \frac{A}{x} + \frac{Bx+C}{x^2+9}\,dx$

$x(x^2+9)$

$$\frac{4x+2}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$4x+2 = A(x^2+9) + x(Bx+C)$$

x^2: $0 = A + B$ $\quad B = -\frac{2}{9}$

x: $4 = C$ $\quad C = 4$

const: $2 = 9A$ $\quad A = \frac{2}{9}$

Break this up!

$$= \int \frac{2/9}{x} + \frac{-2/9\,x + 4}{x^2+9}\,dx$$

$$= \frac{2}{9}\ln|x| - \frac{2}{9}\frac{1}{2}\int \frac{2x\,dx}{x^2+9} + \int \frac{4}{x^2+9}\,dx$$

$$= \frac{2}{9}\ln|x| - \frac{1}{9}\ln|x^2+9| + 4\left(\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right) + C$$

$$\boxed{= \frac{2}{9}\ln|x| - \frac{1}{9}\ln|x^2+9| + \frac{4}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

OR

$$\boxed{= \frac{1}{9}\ln\left(\frac{x^2}{x^2+9}\right) + \frac{4}{3}\tan^{-1}\left(\frac{x}{3}\right) + C}$$

6 (b) $\int \frac{x^3}{x^2+5x}dx = \int \frac{x^3}{x(x+5)}\,dx = \int \frac{x^2}{x+5}\,dx$

$$\begin{array}{r} x - 5 \\ x+5 \enclose{longdiv}{x^2} \\ -(x^2+5x) \\ \hline -5x \\ -(-5x-25) \\ \hline 25 \end{array}$$

$$= \int x - 5 + \frac{25}{x+5}\,dx$$

$$\boxed{= \frac{1}{2}x^2 - 5x + 25\ln|x+5| + C}$$

5. (15 pts) Use the method of your choice to integrate:

$u = \frac{\pi}{x}$ $\quad du = -\frac{\pi}{x^2}\,dx$

(a) $\int_1^2 \frac{1}{x^2}\sin\left(\frac{\pi}{x}\right)dx$

$x = 1 \Rightarrow u = \pi$

$x = 2 \Rightarrow u = \frac{\pi}{2}$

$$= -\frac{1}{\pi}\int_{\pi}^{\pi/2} \sin u \, du$$

$$= -\frac{1}{\pi}\left(-\cos u \Big|_{\pi}^{\pi/2}\right) = -\frac{1}{\pi}\left(-\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} - (-\cos\pi)\right) = \boxed{\frac{1}{\pi}}$$

(b) $\int x^3\cos(2x)\,dx$ $= \boxed{\frac{1}{2}x^3\sin(2x) + \frac{3}{4}x^2\cos(2x) - \frac{3}{4}x\sin(2x) - \frac{3}{8}\cos(2x) + C}$

	u	dv
$+$	x^3	$\cos(2x)$
$-$	$3x^2$	$\frac{1}{2}\sin(2x)$
$+$	$6x$	$-\frac{1}{4}\cos(2x)$
$-$	6	$-\frac{1}{8}\sin(2x)$
$+$	0	$\frac{1}{16}\cos(2x)$

(c) $\int \frac{3x-3}{x^2-2x-3}\,dx$ partial fractions OR u-sub

$(x+1)(x-3)$

partial fractions

$$= \int \frac{A}{x+1} + \frac{B}{x-3}\,dx$$

$$= \int \frac{3/2}{x+1} + \frac{3/2}{x-3}\,dx$$

$$\frac{3x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$3x - 3 = A(x-3) + B(x+1)$

$x:\ 3 = A + B$

const: $-3 = -3A + B$

$-6 = -4A$

$A = \frac{3}{2},\ B = \frac{3}{2}$

$$= \boxed{\frac{3}{2}\ln|x+1| + \frac{3}{2}\ln|x-3| + C}$$

$$= \frac{3}{2}\ln|x^2-2x-3| + C$$

u-sub:

$u = x^2 - 2x - 3$

$du = 2x - 2\,du$

$= 2(x-1)\,du$

$$\int \frac{3(x-1)\,dx}{x^2-2x-3}$$

$$= \frac{3}{2}\int \frac{1}{u}\,du$$

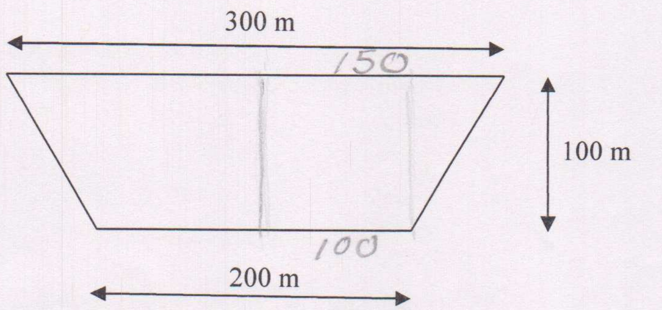
$$= \frac{3}{2}\ln|u| + C$$

$$= \frac{3}{2}\ln|x^2-2x-3| + C$$

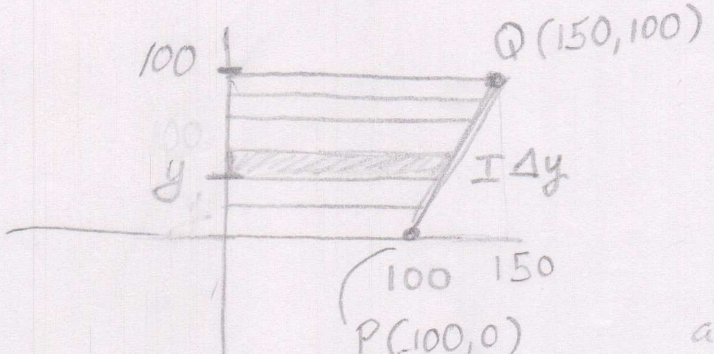
SAME ANSWER

6. (12 pts) Use calculus to find the force that the water in a reservoir exerts on a trapezoidal dam with the following dimensions: height is 100 meters, base is 200 meters, and length of the top edge is 300 meters, which is the top of the dam. Assume the dam is full.

Express your answer in terms of ρ and g.



$$m_{PQ} = \frac{100}{50} = 2$$



line equation for $\overleftrightarrow{PQ}$:

$$y - 0 = 2(x - 100)$$

$$y = 2x - 200$$

$$\Rightarrow x = \frac{y + 200}{2}$$

$$\text{Force} = 2\int_{y=0}^{100} \underbrace{\rho g \underbrace{(100-y)}_{\text{depth}}}_{\text{pressure}} \overbrace{\underbrace{\left(\frac{y+200}{2}\right)}_{\text{width of slice}} dy}^{\text{area of slice}}$$

$$= 2\left(\tfrac{1}{2}\right)\rho g \int_0^{100} (100-y)(y+200)\,dy$$

$$= \rho g \int_0^{100} 20{,}000 - 100y - y^2 \, dy$$

$$= \rho g \left[20{,}000y - 50y^2 - \tfrac{1}{3}y^3\right]_0^{100}$$

$$= \tfrac{1}{3}\rho g (3.5 \times 10^6) \approx 1.14 \times 10^{10}\ N$$

Note:
$\rho = 1000 \frac{kg}{m^3}$
and
$g = 9.8 \frac{m}{s^2}$

7. (18 points) Evaluate each of the following improper integrals or state that it diverges.
You must use the proper notation for full credit.

a) $\int_1^{\infty} \frac{x}{x^2+1}\, dx$

$= \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\, dx$

$\int_1^b \frac{x}{x^2+1}\, dx = \ln|x^2+1|\Big|_1^b$

$= \ln(b^2+1) - \ln(2)$

$= \lim_{b\to\infty} \left[\ln(b^2+1) - \ln(2)\right]$

$= \infty \Rightarrow$ integral <u>diverges</u>

b) $\int_0^{\infty} xe^{-x}\, dx$ (Hint: Integration by Parts)

u	dv
$+\ x$	e^{-x}
$-\ 1$	$-e^{-x}$
$+\ 0$	e^{-x}

$= \lim_{b\to\infty} \int_0^b xe^{-x}\, dx$

$\int_0^b xe^{-x}\, dx$

$= -xe^{-x} - e^{-x}\Big|_0^b$

$= -be^{-b} - e^{-b} - (0 - e^0)$

$= 1 - \frac{b}{e^b} - \frac{1}{e^b}$

$= \lim_{b\to\infty} \left[1 - \frac{b}{e^b} - \frac{1}{e^b}\right]$

$= 1 - 0 - 0$

$\boxed{= 1}$

Extra credit (5 points)

Integrate, using any method. Use the back of this page for your work.

$\int \frac{x^3 e^{\sqrt{4-x^2}}}{\sqrt{4-x^2}}\, dx$

$$\int \frac{x^3 e^{\sqrt{4-x^2}}}{\sqrt{4-x^2}}\,dx$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

so $\sqrt{4-4\sin^2\theta}$

$= 2\sqrt{1-\sin^2\theta}$

$= 2\sqrt{\cos^2\theta}$

$= 2\cos\theta$

$$= \int \frac{8\sin^3\theta\, e^{2\cos\theta}\cdot \cancel{2\cos\theta}\,d\theta}{\cancel{2\cos\theta}}$$

$$= -\int 4\sin^2\theta \cdot e^{2\cos\theta}(-2\sin\theta)\,d\theta$$

Let $w = 2\cos\theta$

$dw = -2\sin\theta\,d\theta$

$w^2 = 4\cos^2\theta$

$$= -\int \overbrace{4(1-\cos^2\theta)}^{4-4\cos^2\theta}\, e^{2\cos\theta}(-2\sin\theta)\,d\theta$$

$$= -\int (4-w^2)e^w\,dw$$

$$= \int (w^2-4)e^w\,dw$$

PARTS

u	dv
$(+)\ w^2-4$	e^w
$(-)\ 2w$	e^w
$(+)\ 2$	e^w
0	e^w

$$= (w^2-4)e^w - 2we^w + 2e^w + C$$

$$= w^2e^w - 2we^w - 2e^w + C$$

$$= 4\cos^2\theta\, e^{2\cos\theta} - 4\cos\theta\, e^{2\cos\theta} - 2e^{2\cos\theta} + C$$

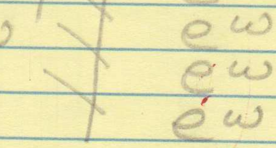
$$= 4\left(\frac{\sqrt{4-x^2}}{2}\right)^2 e^{\sqrt{4-x^2}} - 4\left(\frac{\sqrt{4-x^2}}{2}\right)e^{\sqrt{4-x^2}} - 2e^{\sqrt{4-x^2}} + C$$

$$= e^{\sqrt{4-x^2}}\left(4-x^2-2\sqrt{4-x^2}-2\right) + C$$

$$= e^{\sqrt{4-x^2}}\left(2-x^2-2\sqrt{4-x^2}\right) + C$$

$x = 2\sin\theta$

$\Rightarrow \frac{x}{2} = \sin\theta$



so $\cos\theta = \frac{\sqrt{4-x^2}}{2}$