Math 265B: Test 2 **Name:** KEY

(100 points)

Credit for each problem is based on the amount of correct work shown, not just the final answer Simplify all answers as much as possible but for full credit, leave answers in exact terms. Only scientific calculators may be used on this exam.

Formulas: $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$ $\cos^2\phi = \frac{1}{2}(1+\cos 2\phi)$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

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1. (8 pts) Given the integral $\int \frac{x^4}{\sqrt{x^2+16}}\,dx$, do not attempt to integrate but answer the following questions

2 (a) Could you use Partial Fraction Decomposition to integrate? YES NO Why or why not?

The square root means the integrand is not a rational function.

3 (b) Could you use Trig Substitution? $\boxed{\text{YES}}$ NO If yes, what would the substitution be for x and dx?

Yes. $x = 4\tan\theta$

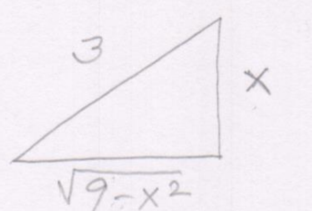
$dx = 4\sec^2\theta\, d\theta$

2 (c) Could you use u-Substitution (or Reverse the Chain Rule/Guess-and-Check)? YES $\boxed{\text{NO}}$ If yes, what would u be?

1 (d) Could you manipulate the integrand to be a simpler function? YES $\boxed{\text{NO}}$
If yes, show what you would do:

2. (4 pts) Suppose you made a Trig Substitution of $x = 3\sin\theta$ in an integral (this is unrelated to problem #1)

Which of the following would be true? *Show work or a sketch for credit!* $\sin\theta = \frac{x}{3}$



a) $\cos\theta = \frac{\sqrt{9-x^2}}{3}$ b) $\cot\theta = \frac{\sqrt{9-x^2}}{x}$

c) both of these d) neither of these

3. (15 pts) Determine which of the integration techniques listed below would be the ***BEST*** to apply to find the following integrals. You do not have to integrate.

Techniques:
Partial Fraction Decomposition, Integration by Parts, u-Substitution, Trig Substitution,

Half Angle Trig Identity, Pythagorean Trig Identify

a.) $\int x\sin(x)\,dx$ Best Technique: Integration by Parts

b.) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ Best Technique: u-Substitution

c.) $\int \frac{1}{x^3+x}\,dx$ Best Technique: Partial Fractions

d.) $\int \cos^2(x)\sin^2(x)dx$ Best Technique: Half-Angle Trig Identity

e.) $\int \frac{1}{\sqrt{x^2-25}}dx$ Best Technique: Trig Substitution

4. (10 pts) Integrate. Suggested Method: Pythagorean Identity.

$$\int \cos^3 x\left(\sin x\right)^{1/2} dx$$

$u = \sin x$
$du = \cos x\,dx$

$$= \int \cos^2 x \cdot \cos x \cdot (\sin x)^{1/2}\,dx$$

$$= \int (1-\sin^2 x)(\sin x)^{1/2}\cos x\,dx$$

$$= \int (1-u^2)\,u^{1/2}\,du$$

$$= \int u^{1/2} - u^{5/2}\,du$$

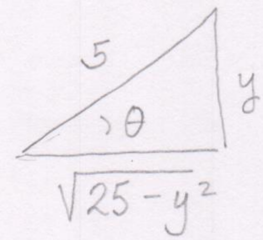
$$= \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2} + C$$

$$\boxed{= \frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + C}$$

5. (10 pts) Integrate: Suggested method: Trig Substitution (Hint: $\int \csc^2\theta\, d\theta = -\cot\theta + C$)

$$\int_3^4 \frac{1}{y^2\sqrt{25-y^2}}\,dy$$

$$y = 5\sin\theta$$
$$dy = 5\cos\theta\, d\theta$$



$$\cot\theta = \frac{\sqrt{25-y^2}}{y}$$

$$= \int_{y=3}^{y=4} \frac{\cancel{5}\cos\theta\, d\theta}{\cancel{25}\sin^2\theta\sqrt{25-25\sin^2\theta}}$$

$$= \frac{1}{5}\int_{y=3}^{y=4} \frac{\cos\theta\, d\theta}{\sin^2\theta\cdot\sqrt{25(1-\sin^2\theta)}}$$

$$= \frac{1}{25}\int_{y=3}^{y=4} \frac{\cancel{\cos\theta}\, d\theta}{\sin^2\theta\cdot\cancel{\cos\theta}}$$

$$= \frac{1}{25}\int_{y=3}^{y=4} \csc^2\theta\, d\theta$$

$$= \frac{1}{25}\left[-\cot\theta\right]_{y=3}^{y=4}$$

$$= -\frac{1}{25}\,\frac{\sqrt{25-y^2}}{y}\Bigg|_3^4$$

$$= -\frac{1}{25}\left[\frac{3}{4}-\frac{4}{3}\right]$$

$$\boxed{= \frac{7}{300}}$$

6. (12 pts) Integrate. Suggested method: Partial Fraction Decomposition.

$$\int \frac{2x+4}{x^2-2x-3}\,dx.$$

$(x-3)(x+1)$

$$\frac{2x+4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

$$2x+4 = A(x+1) + B(x-3)$$

x coef: $(2 = A + B)\cdot 3$

const: $4 = A - 3B$

$$10 = 4A$$

$$A = \frac{10}{4} = \frac{5}{2}$$

$$B = 2 - \frac{5}{2} = -\frac{1}{2}$$

$$= \int \frac{5/2}{x-3} + \frac{-1/2}{x+1}\,dx$$

$$\boxed{= \frac{5}{2}\ln|x-3| - \frac{1}{2}\ln|x+1| + C}$$ ok

$$= \frac{1}{2}\ln\left|\frac{(x-3)^5}{x+1}\right| + C$$

any of these forms is fine.

$$= \ln\left|\frac{(x-3)^5}{x+1}\right|^{1/2} + C$$

7. (12 pts) Integrate. Suggested method: Integration by Parts.

$$\int_0^4 x^2 e^{-5x}\,dx.$$

	u	dv
(+)	x^2	e^{-5x}
(−)	$2x$	$-\frac{1}{5}e^{-5x}$
(+)	2	$+\frac{1}{25}e^{-5x}$
(−)	0	$-\frac{1}{125}e^{-5x}$

$$= -\frac{1}{5}x^2e^{-5x} - \frac{2}{25}xe^{-5x} - \frac{2}{125}e^{-5x}\Big|_0^4$$

$$= -e^{-5x}\left[\frac{1}{5}x^2 + \frac{2}{25}x + \frac{2}{125}\right|_0^4$$

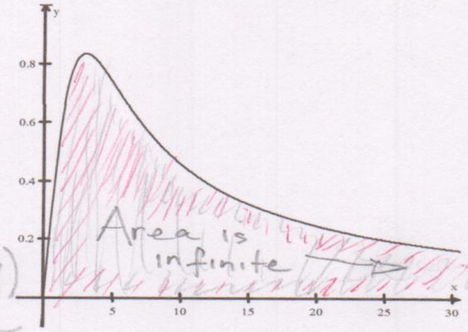
$$= -e^{-20}\left[\frac{442}{125}\right] + e^0\left[\frac{2}{125}\right]$$

$$= \frac{2}{225} - e^{-20}\left[\frac{442}{125}\right]$$

$$\approx .016$$

8. (12 points) Evaluate each of the following improper integrals and state whether it converges or diverges. *You must use the proper limit notation for full credit*. Graphs are provided for reference.

a) $\displaystyle\int_0^\infty \frac{5x}{x^2+9}\,dx$



$$= \lim_{b\to\infty} \int_0^b \frac{5x}{x^2+9}\,dx \quad (*)$$

$$(*)\ \int_0^b \frac{5x}{x^2+9}\,dx$$

$$= \frac{5}{2}\ln|x^2+9|\Big|_0^b$$

$$= \frac{5}{2}\ln(b^2+9) - \frac{5}{2}\ln(9)$$

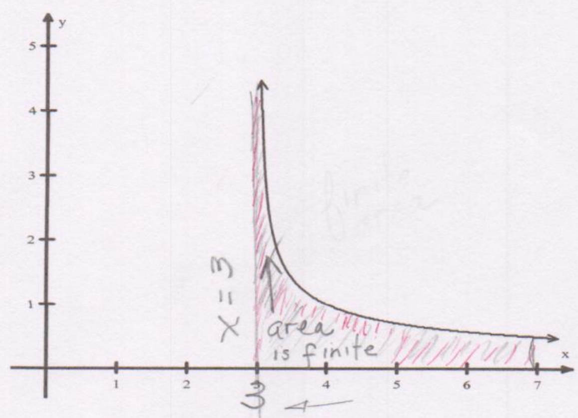
$$(*)$$

$$= \lim_{b\to\infty}\left[\frac{5}{2}\ln(b^2+9) - \frac{5}{2}\ln(9)\right]$$

$$= \infty$$

The integral diverges

b) $\displaystyle\int_3^7 \frac{1}{(x-3)^{1/2}}\,dx$



$$= \lim_{b\to 3^+} \int_b^7 (x-3)^{-\frac{1}{2}}\,dx \quad (*)$$

$$(*)\ \int_b^7 (x-3)^{-1/2}\,dx$$

$$= 2(x-3)^{1/2}\Big|_b^7$$

$$= 2\left[(7-3)^{1/2} - (b-3)^{1/2}\right]$$

$$= 2\left[2 - (b-3)^{1/2}\right]$$

$$(*)$$

$$= \lim_{b\to 3^+}\left[4 - 2(b-3)^{1/2}\right]$$

$$= 4 - 0 = \boxed{4}$$

The integral converges

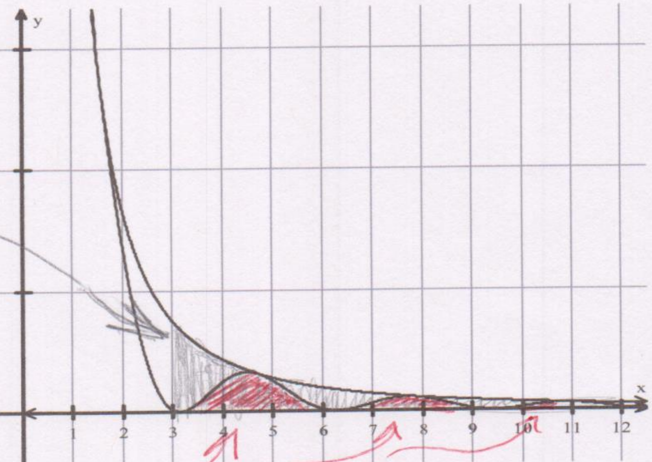
9 (6 pts) (a) Evaluate the integral and state whether it converges or diverges. You may use any type of notation.

$$\int_{\pi}^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{\pi}^{\infty} = 0 + \frac{1}{\pi} = \boxed{\frac{1}{\pi}}$$

The integral converges

$$\int \frac{1}{x^2} = \int x^{-2}dx = -x^{-1} + C = -\frac{1}{x} + C$$

(b) Based on the result from part (a), and the given graphs of $\frac{1}{x^2}$ and $\frac{\sin^2(x)}{x^2}$, briefly explain how you can tell whether the integral $\int_{\pi}^{\infty} \frac{\sin^2(x)}{x^2}dx$ converges or diverges.



The area under $\frac{1}{x^2}$, from $x = \pi$ to ∞, is finite since $\int_{\pi}^{\infty} \frac{1}{x^2}dx$ converges. Since $\frac{\sin^2 x}{x^2}$ is under the $\frac{1}{x^2}$ curve, the area under $\frac{\sin^2 x}{x^2}$ must also be finite.

Applications: **Choose 1 problem below** (10 pts)

Choice #1 for problem 10. Set up and evaluate an improper integral to determine the amount of work it takes for a rocket of mass m to escape the gravity of the Earth, whose mass is M and whose radius is R. You may use any type of notation. Express your answer in terms of G, M, and m. and R

The force of gravity is $F(x) = \frac{GMm}{x^2}$, G = the gravitational constant, x = distance from rocket to center of Earth.

$$\text{Work} = \int_R^{\infty} \frac{GMm}{x^2}dx = GMm\int_R^{\infty}\frac{1}{x^2}dx$$

$$= GMm\left[-\frac{1}{x}\right]_R^{\infty}$$

$$= GMm\left[0 + \frac{1}{R}\right] = \frac{GMm}{R}$$

$$\boxed{\text{Work} = \frac{GMm}{R}}$$

Choice #2 for problem 10. Find the volume of the solid that is generated when the region bounded by $f(x)=\sqrt{x\ln x}$ is revolved about the x-axis on $[1,4]$ (Graph is provided)

$$V=\int_1^4 \pi\left(\sqrt{x\ln x}\right)^2 dx$$

$$=\pi\int_1^4 x\ln x\,dx \quad \star$$

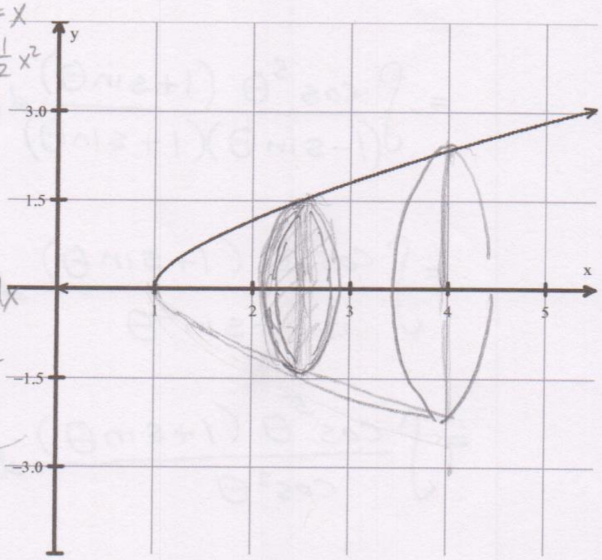
$u=\ln x \quad dv=x$

$du=\frac{1}{x} \quad v=\frac{1}{2}x^2$

$\star \int x\ln x\,dx$

$=\frac{1}{2}x^2\ln x-\int\frac{1}{2}x\,dx$

$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2$



$$=\pi\left[\frac{1}{2}x^2\ln x-\frac{1}{4}x^2\right]_1^4$$

$$=\pi\left[8\ln 4-4-\left(\tfrac{1}{2}\cancel{\ln 1}^{\,0}-\tfrac{1}{4}\right)\right]$$

$$\boxed{=\pi\left[8\ln 4-3.75\right]}$$

Extra credit (4 points)

Integrate, using any method. $\displaystyle\int \frac{\cos^5\theta}{1-\sin\theta}\,d\theta$ Do your work on the back of this sheet.

$$\int \frac{\cos^5\theta}{1-\sin\theta}\,d\theta$$

$$= \int \frac{\cos^5\theta\,(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}\,d\theta$$

$$= \int \frac{\cos^5\theta\,(1+\sin\theta)}{1-\sin^2\theta}\,d\theta$$

$$= \int \frac{\cos^5\theta\,(1+\sin\theta)}{\cos^2\theta}\,d\theta$$

$$= \int \cos^3\theta\,(1+\sin\theta)\,d\theta$$

$$= \int \underbrace{\cos^3\theta}_{\cos^2\theta\ \cos\theta}\,d\theta + \int \cos^3\theta\,\sin\theta\,d\theta$$

$$= \int (1-\sin^2\theta)\cos\theta\,d\theta + \int \cos^3\theta\,\sin\theta\,d\theta$$

$$\boxed{= \sin\theta - \frac{1}{3}\sin^3\theta - \frac{1}{4}\cos^4\theta + C}$$

Note: an alternate form of the answer is $\boxed{-\frac{1}{4}\sin^4\theta - \frac{1}{3}\sin^3\theta + \frac{1}{2}\sin^2\theta + \sin\theta + C}$