Math 265B: Test 2 Take Home (20 pts) Name: KEY

Due: Thursday, at the beginning of class.

Guidelines:

- *You are welcome to work with other students in the class but the final work you hand in must be your own. Your answers must match every step of your work; otherwise, you may lose most or all of the points for the problem. Do all work by hand; i.e., if you use Wolfram or another CAS to check your work (encouraged!) you still need to show your own work.*
- *Please do not ask Math Lab tutors (or anyone else) to solve the problems for you.*
- ***Do your work on separate paper*** *and attach this page as a cover sheet. Make sure your work is clear, legible and well organized. Work that is poorly organized and/or difficult to read will be marked down*

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1. (4 points) Evaluate each of the following improper integrals and state whether it converges or diverges.
You must use the proper limit notation for full credit.

a) $\displaystyle\int_0^{\infty} \frac{5}{x^2+1}\,dx$ b) $\displaystyle\int_5^{9} \frac{1}{(x-5)^{1/2}}\,dx$

2 (6 pts) Find the volume of the solid that is formed by revolving $y = \dfrac{1}{\sqrt{x^2+1}}$ about the x-axis, over the interval $[0,\infty)$. Sketch a clear illustration of the graph and the solid formed by revolution.

3. (10 pts) The force of gravity between an object of mass M and a second object of mass m, separated by a distance r, is given by $F = \dfrac{GMm}{r^2}$ (Inverse Square Law)

(a) Determine the work needed to move an object of mass m out of the influence of Earth's gravity (one of the limits of integration should be infinity!). Leave your answer in terms of m, but use the following values (suggestion: Do the integration first, then substitute!)

$M_{Earth} = 5.972 \times 10^{24}\,kg$

$r_{Earth} = 6370\ km$ (mind units!) convert to meters!

G = 6.67259 x 10^{-11} N m^2/kg^2

(b) What velocity would the object need to have in order to have the energy to completely escape from the Earth's gravitational pull? Kinetic energy, which is the energy of objects in motion, is $\frac{1}{2}mv^2$.

Take Home: KEY

1 (a) $$\int_0^\infty \frac{5}{x^2+1}\,dx = \lim_{b\to\infty}\left[\int_0^b \frac{5}{x^2+1}\,dx\right]$$

let's track where this 5 goes (You didn't have to do this!)

$$= \lim_{b\to\infty}\left[5\cdot\int_0^b \frac{1\,dx}{x^2+1}\right]$$

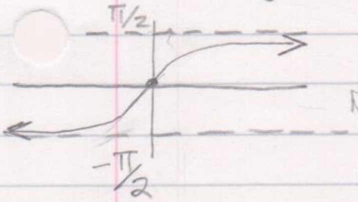
(5 moves out of integral (WHY?)

$$= 5\cdot\lim_{b\to\infty}\left[\int_0^b \frac{1\,dx}{x^2+1}\right]$$

(5 moves out of limit (limit prop.) (WHY?)

$$= 5\cdot\lim_{b\to\infty}\left[\tan^{-1}(x)\Big|_0^b\right]$$

(*) $\tan^{-1}$ graph



$$= 5\cdot\lim_{b\to\infty}\left[\tan^{-1}(b) - \cancel{\tan^{-1}(0)}^{\,0}\right]$$

$$\overset{(*)}{=} 5\cdot\frac{\pi}{2} = \boxed{\frac{5\pi}{2}} \quad \boxed{\text{Converges!}}$$

Note: This is a pretty simple problem, especially for a Take Home, so it gives you enough breathing room to think about WHY* for each step.

* Super helpful in getting a more complete understanding of how all these pieces work separately and how they fit together! :)

1 (b) $\int_5^9 \frac{1}{(x-5)^{1/2}}\,dx = \lim_{b\to 5^+}\left[\int_b^9 \frac{1}{(x-5)^{1/2}}\,dx\right]$ evaluate separately (see below) ☆

singularity at $x=5$

$= \lim_{b\to 5^+}\left[4-2\sqrt{b-5}\right]$

$= 4-2(0)^{1/2}$

$\boxed{=4}$

$\boxed{\text{Converges!}}$

☆ $\int_{x=b}^{x=9} \frac{1}{(x-5)^{1/2}}\,dx$ $\quad u=x-5$, $du=dx$

$= \int_{x=b}^{x=9} \frac{1}{u^{1/2}}\,du$

$= \int_{x=b}^{x=9} u^{-\frac{1}{2}}\,du$

$= 2u^{\frac{1}{2}}\Big|_{x=b}^{x=9}$

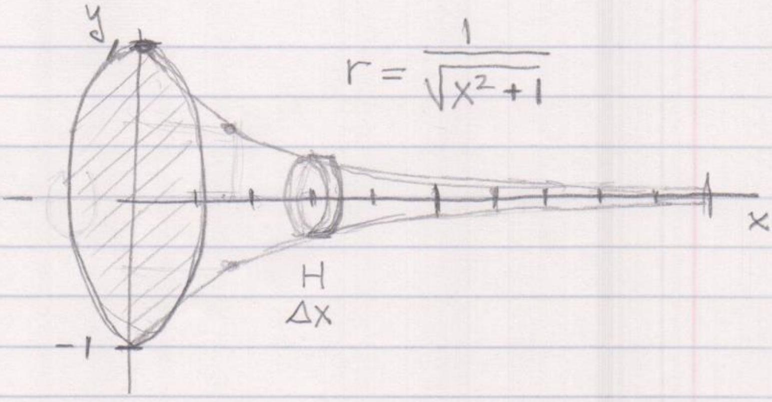
$= 2\sqrt{x-5}\,\Big|_b^9$

$= 2\sqrt{4} - 2\sqrt{b-5}$

$= 4-2\sqrt{b-5}$

2. $y = \dfrac{1}{\sqrt{x^2+1}}$

revolve about x-axis



$r = \dfrac{1}{\sqrt{x^2+1}}$

$$V = \int_0^{\infty} \pi \left(\frac{1}{\sqrt{x^2+1}}\right)^2 dx$$

$$= \int_0^{\infty} \pi \frac{1}{x^2+1}\, dx$$

$$= \lim_{b \to \infty} \pi \int_0^b \frac{1}{x^2+1}\, dx$$

$$\int_0^b \frac{1}{x^2+1}\, dx$$

$$= \tan^{-1}(x) \Big|_0^b$$

$$= \tan^{-1}(b) - \cancel{\tan^{-1}(0)}\ 0$$

$$= \pi \lim_{b \to \infty} \tan^{-1}(b)$$

$$= \pi\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi^2}{2} \text{ unit}^3}$$

3 (a) Work = Force · distance $= \int F(r)\, dr$

$$\text{Work} = \int_{r_E}^{\infty} \frac{GMm}{r^2}\, dr$$

$$= \lim_{b \to \infty} \int_{r_E}^{b} \frac{GMm}{r^2}\, dr$$

$$\int_{r_E}^{b} \frac{GMm}{r^2}\, dr$$

$$= -\frac{GMm}{r} \Bigg|_{r_E}^{b}$$

$$= -\frac{GMm}{b} + \frac{GMm}{r_E}$$

$$= \lim_{b \to \infty} \left[\frac{GMm}{r_E} - \frac{GMm}{b}\right]$$

$$= \frac{GMm}{r_E} - 0 = \boxed{GMm/r_E}$$

constant! limit of a constant is a constant!

3 (a) continued $r_E = 6370$ km $= 6{,}370{,}000$ m

$$\text{Work} = \frac{GMm}{r_E} = \frac{6.67259\times10^{-11} \cdot 5.972\times10^{24} \cdot m}{6.37\times10^{6}}$$

$$= (6.2559 \times 10^{7})\, m \quad \text{N-m (or Joules)}$$

mass (not meters)

(b) Work = Energy

$(6.2559 \times 10^{7})\, m_o = \frac{1}{2} m_o v_e^2$ m_o = mass of object (kg)

(work to move object out of Earth's gravity) (kinetic energy of moving object) v_e = escape velocity (m/s)

$$6.2559\times10^{7} = \frac{1}{2} v_e^2$$

$$v_e = \sqrt{2 \times 6.2559 \times 10^{7}}$$

$= 11{,}186$ m/s

≈ 11 km/s (about 24,500 mph!)