Math 265B: Test 3 Name: KEY

For maximum credit, please show all of your work in a clear, logical manner. Simplify all answers as much as possible. Leave answers in exact form unless an approximation is asked for in the instructions.

1. (15 points) Evaluate each of the following improper integrals and state whether it converges or diverges. You must use the proper notation for full credit.

a) $\int_0^\infty \frac{x}{x^2+1}\,dx = \lim_{B\to\infty} \int_0^B \frac{x}{x^2+1}\,dx$

$\int_0^B \frac{x}{x^2+1}\,dx$

$= \frac{1}{2}\ln|x^2+1| \Big|_0^B$

$= \frac{1}{2}\ln(B^2) - \cancel{\frac{1}{2}\ln(1)} \to 0$

$= \frac{1}{2}\ln(B^2)$

$= \lim_{B\to\infty}\left[\frac{1}{2}\ln B^2\right] = \infty$

$\boxed{\text{Diverges}}$

b) $\int_{\frac{1}{\pi}}^{\infty} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$

$= \lim_{B\to\infty} \int_{1/\pi}^{B} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$

$\int_{1/\pi}^{B} \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$

$= \cos\left(\frac{1}{x}\right)\Big|_{1/\pi}^{B}$

$= \cos\left(\frac{1}{B}\right) - \cos(\pi)$

$= \cos\left(\frac{1}{B}\right) + 1$

$= \lim_{B\to\infty}\left[\cos\left(\frac{1}{B}\right)+1\right]$

$= \cos(0) + 1 = \boxed{2}$

$\boxed{\text{Converges}}$

c) $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$

Bad value! Division by zero!

$= \lim_{B\to 0^+} \int_B^8 x^{-1/3}\,dx$

$\int_B^8 x^{-\frac{1}{3}}\,dx$

$= \frac{3}{2}x^{\frac{2}{3}}\Big|_B^8$

$= \frac{3}{2}\cdot 8^{\frac{2}{3}} - \frac{3}{2}B^{\frac{2}{3}}$

$= 6 - \frac{3}{2}B^{2/3}$

$= \lim_{B\to 0^+}\left[6 - \frac{3}{2}B^{2/3}\right]$

$= \boxed{6}$

so $\boxed{\text{converges}}$

2. (6 pts) Determine whether the following integrals converge or diverge. Explain how you are determining this. You do not have to integrate!

a) $\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx$ DIVERGES

p-test with $p = \frac{1}{2} \leq 1$

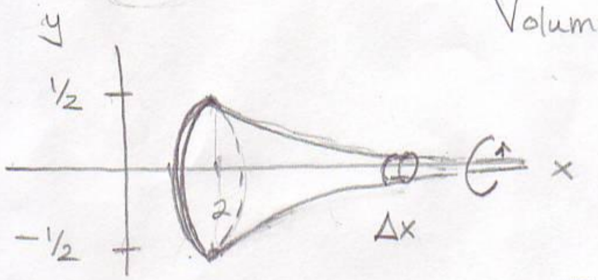
b) $\int_1^{\infty} \underbrace{\frac{x^2-2}{x^6+5x}}\,dx$ CONVERGES

Behaves like $\frac{1}{x^4}$ as $x \to \infty$

so by p-test, with $p = 4 > 1$

3. (6 pts) Find the volume of the solid of revolution formed by spinning $y = \frac{1}{x}$, $2 \leq x < \infty$ about the x-axis.

Sketch the solid of revolution.



$$\text{Volume} = \int_2^{\infty} \pi \left(\frac{1}{x}\right)^2 dx$$

$$= \lim_{B\to\infty} \int_2^B \pi x^{-2}\,dx$$

$$\int_2^B \pi x^{-2}\,dx$$

$$= -\frac{\pi}{x}\Big|_2^B$$

$$= -\frac{\pi}{B} + \frac{\pi}{2}$$

$$= \lim_{B\to\infty} \left[-\frac{\pi}{B} + \frac{\pi}{2}\right]$$

$$\boxed{= \frac{\pi}{2}\ \text{unit}^3}$$

4. (6 pts) Determine which (one, both, or neither) of the following functions is a solution to the differential equation $y''(t) + 4y(t) = 0$. Show work that justifies your answer.

(a) $y(t) = e^{-2t}$

$y'(t) = -2e^{-2t}$

$y''(t) = 4e^{-2t}$

$y'' + 4y = 0$?

$4e^{-2t} + 4e^{-2t} = 0$

$8e^{-2t} = 0$ false!

Solution? Yes No

(b) $y(t) = \cos(2t)$

$y'(t) = -2\sin(2t)$

$y''(t) = -4\cos(2t)$

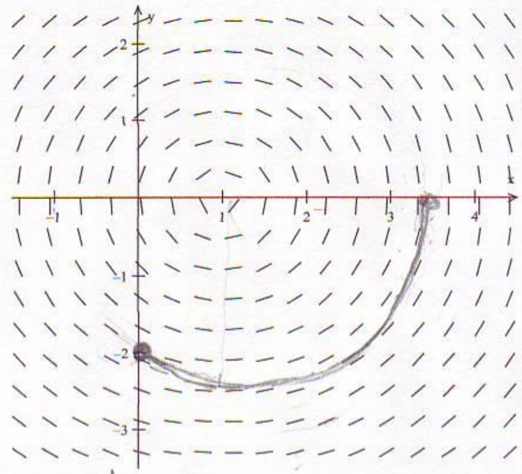
$-4\cos(2t) + 4\cos(2t) \overset{?}{=} 0$

$0 = 0$ true!

Solution? Yes No

5. (8 pts) Find the solution to the following Initial Value Problem analytically. Then on the slope field graph provided, sketch the solution function, $y(t)$, $t \geq 0$

$$y' = \frac{1-t}{y}, \qquad y(0) = -2$$



$y\,dy = 1 - t\,dt$

$\frac{1}{2}y^2 = t - \frac{1}{2}t^2 + C$

$y^2 = 2t - t^2 + C_1$

$y = \pm\sqrt{2t - t^2 + C_1}$

$t=0$

Since $y(0) = -2$, we want the lower half of the circle, so

$y(t) = \sqrt{2t - t^2 + C_1}$ $\Longrightarrow$ Evaluate C_1: $y(0) = -2$

$y(0) = -\sqrt{2\cdot 0 - 0^2 + C_1}$

$\sqrt{C_1} = 2$

$C_1 = 4$

$\boxed{y(t) = -\sqrt{2t - t^2 + 4}}$

6. (6 pts) Consider the Initial Value Problem $\dfrac{dy}{dx} = (y-2)(7-y), \quad y(0) = y_0$.

(a) Find any equilibrium solutions and classify them as stable or unstable.

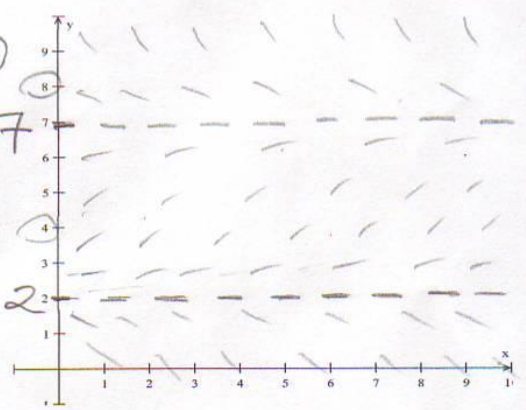
unstable $y = 2$ $(y(t) = 2)$

stable $y = 7$ $(y(t) = 7)$

$y'\big|_{y=8} = \ominus$

$y'\big|_{y=4} = \oplus$

$y'\big|_{y=0} = \ominus$



(b) Sketch a slope field for the differential equation.

(c) For what initial conditions (i.e., what values of y_0) will the solution curves be increasing functions?

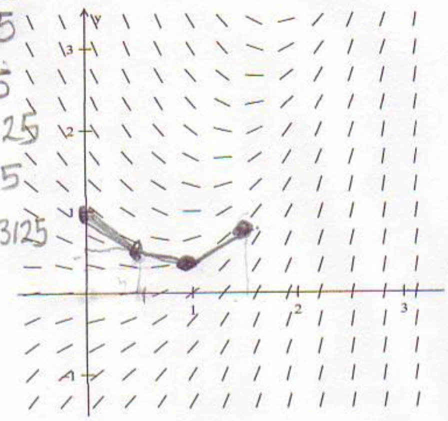
For $2 < y_0 < 7$ (where y' is positive)

7. (12 pts) Consider the I.V.P. $\frac{dy}{dt} = t^2 - y, \quad y(0) = 1$

with $\Delta t = .5$
(we'll need 3 steps

(a) Use Euler's Method, with $\Delta t = 0.5$ to estimate the value of $y(t)$, for $t = 1.5$ Clearly show all of your work.

(t_K, y_K)	$y' = t^2 - y$	$\Delta y = m\Delta t$	$x_{K+1} = x_K + \Delta t$ $y_{K+1} = y_K + \Delta y$
$(0, 1)$	$y' = 0^2 - 1 = -1$	$\Delta y = -1(.5) = -.5$	$y_{K+1} = 1 - .5 = +.5$
$(.5, .5)$	$y' = .5^2 - .5 = -.25$	$\Delta y = -.25(.5) = -.125$	$y_{K+1} = .5 - .125 = .375$
$(1, .375)$	$y' = 1^2 - .375 = .625$	$\Delta y = .625(.5) = .3125$	$y_{K+1} = .375 + .3125 = .6875$
$(1.5, 6.875)$			



$y(1.5) \cong$ .6875

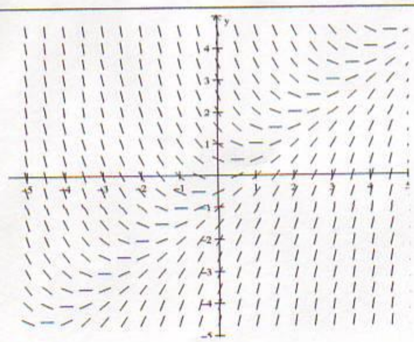
(b) On the slope field provided, sketch the polygonal Euler's solution "curve", based on your work from part (a).

8. (8 pts) Label the slope fields with the one differential equation that best matches it. No work need be shown.

(i) $y' = t - y$ (ii) $y' = -y^2 t$ (iii) $y' = (y-3)\ln y$ (iv) $y' = \cos t + \cos y$

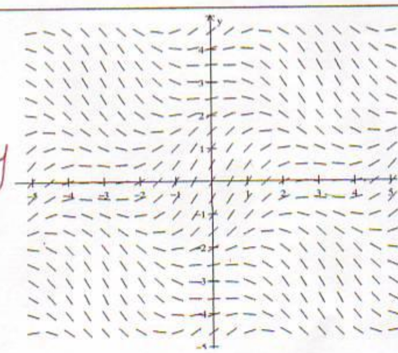
(a) $\boxed{i}$

$y' = t - y$



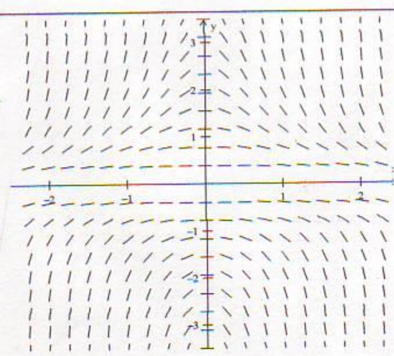
(b) $\boxed{IV}$

$y' = \cos t + \cos y$



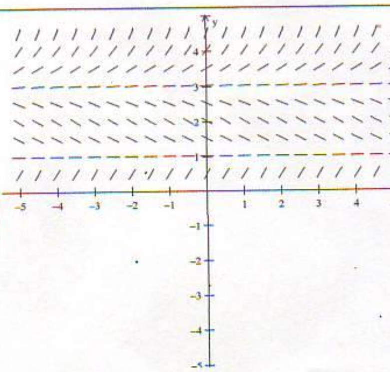
(c) $\boxed{ii}$

$y' = -y^2 t$



(d) $\boxed{iii}$

$y' = (y-3)\ln y$



9. (4 pts) Which of the following differential equations are separable? (Circle those that are.)

(i) $y' = t^2 - y^2$ (ii) $\boxed{y' = y^2 - y^2 t}$ $y^2(1-t)$ (iii) $\boxed{y' = (y-3)\ln y}$ (iv) $y' = \cos t + \cos y$

10. (8 pts) Sierra ran over a goat head thorn and now has a slow leak in her bike tire. Assume that the pressure in the tire decreases at a rate proportional to the difference between the atmospheric pressure (take this to be 14 psi) and the tire pressure. Assume the tire pressure was initially 35 psi.

(a) Let p(t) be the pressure in the tire t hours since Sarah ran over the thorn. Set up an Initial Value Problem that models this situation. Leave k (the constant of variation) as a general constant.

$$\frac{dp}{dt} = k(14-p)$$

$$p(0) = 35$$

(b) Solve the I.V.P. Again, leave k as a general constant. Show the steps in finding the constant, C_1.

$$\int \frac{dp}{p-14} = \int -k\,dt$$

$$e^{\ln|p-14|} = e^{-kt+c}$$

$$p-14 = \pm e^c e^{-kt}$$

$$p(t) = 14 + c_1 e^{-kt}$$

Evaluate C_1:

$$p(0) = 35$$

$$35 = 14 + c_1 e^0$$

$$C_1 = 21$$

so $\boxed{p(t) = 14 + 21e^{-kt}}$

(c) What is the steady state for the solution function you found in part (b)? How does that relate to the equilibrium solution for the original differential equation and what does it mean in terms of Sarah's tire?

Steady state:

$$y = \lim_{t\to\infty} 14 + 21e^{-kt} = 14$$

This is the equilibrium solution.
Over time the pressure in Sarah's tire will be the same as the atmospheric pressure.

11. (8 pts) Given that A(t) is the amount of a substance in a compartment at time, t, the compartmental analysis model is as follows:

$$\begin{matrix}\textit{Rate of change}\\ \textit{of amount in compartment}\end{matrix} = \begin{pmatrix}\textit{concentration}\\ \textit{going into compartment}\end{pmatrix}\cdot\begin{pmatrix}\textit{inflow}\\ \textit{rate}\end{pmatrix} - \begin{pmatrix}\textit{concentration}\\ \text{out of } \textit{compartment}\end{pmatrix}\cdot\begin{pmatrix}\textit{outflow}\\ \textit{rate}\end{pmatrix}$$

where $\begin{pmatrix}\textit{concentration}\\ \textit{out of compartment}\end{pmatrix} = \dfrac{A(t)}{\textit{volume of compartment}}$

Apply this model to set up and solve this tank mixture problem:

A 2000-L tank is filled with a brine (salt) solution with an initial concentration of 4 g/L. Brine solution with a concentration of 20g/L flows into the tank at a rate of 5 L/min, while thoroughly mixed solution flows out of the tank at 5 L/min. Find the function A(t) = the amount of salt in the tank after t minutes.

$$\frac{dA}{dt} = 20\frac{g}{L}\cdot 5\frac{L}{min} - \frac{A}{2000}\frac{g}{L}\cdot 5\frac{L}{min}$$

$$\frac{dA}{dt} = 100 - \frac{1}{400}A = -\frac{1}{400}\left[A - 40{,}000\right]$$

$$\int \frac{dA}{A-40000} = \int -\frac{1}{400}\,dt$$

$$e^{\ln|A-40{,}000|} = e^{-\frac{1}{400}t + C}$$

$$A = 40{,}000 + C_1 e^{-\frac{1}{400}t}$$

$A(0) = 4\frac{g}{L}\cdot 2000\text{ L}$

$A(0) = 8000\text{ g}$

* (note: we could convert to kilograms to get rid of all these zeros!)

evaluate C_1

$\begin{cases} A(0) = 8000 \\ A(0) = 40{,}000 + C_1 e^0 \end{cases}$

$C_1 = -32\,000$

$$\boxed{A(t) = 40000 - 32000e^{-\frac{1}{400}t}\text{ g}}$$

$$* \ A(t) = 40 - 32e^{-1/400\,t}\text{ kg}$$

12. (8 pts) Use the Midpoint and Trapezoid Rules, with n = 2, to estimate the value of the definite integral. Illustrate your work on the graphs provided. Give your answers in exact terms and as approximations to 2 decimal places.

Midpoint Approximation:

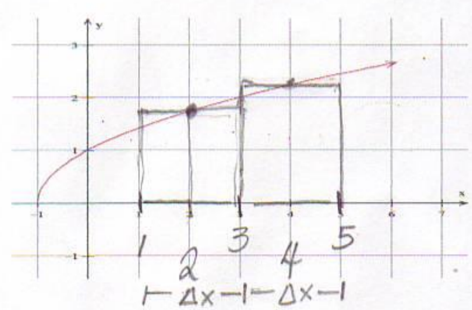
$$\int_1^5 \sqrt{x+1}\,dx \approx f(2)\Delta x + f(4)\Delta x$$

$$= \sqrt{3}\cdot 2 + \sqrt{5}\cdot 2$$

$f(x) = \sqrt{x+1}$

$$\text{OR } = 2(\sqrt{3} + \sqrt{5})$$

$$\approx 7.94 \text{ (about 8 "boxes"!)}$$

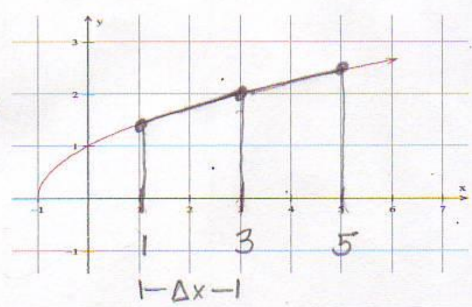


Trapezoid Approximation:

$$\int_1^5 \sqrt{x+1}\,dx \approx \frac{1}{2}\left[f(1) + f(3)\right]\Delta x + \frac{1}{2}\left[f(3) + f(5)\right]\Delta x$$

$$= \frac{\sqrt{2} + \sqrt{4}}{2}\cdot 2 + \frac{\sqrt{4} + \sqrt{6}}{2}\cdot 2$$

$$= \sqrt{2} + 4 + \sqrt{6} \approx 7.86$$



Extra Credit (4 points) Suppose that for a certain definite integral, TRAP(10) = 12.676 and TRAP(30) = 10.420. Use this information to estimate the true value of the integral.

Recall that for the trapezoid rule, $Error(kn) \cong \dfrac{Error(n)}{k^2}$

(Hint: You'll have to set up and solve a simple system of equations.)

Let $A = \int_a^b f(x)dx$ ("A" for "actual value")

we know the actual value is the estimated value plus the error.

So $A = \text{Trap}(10) + \text{Error}(10)$

and $A = \text{Trap}(30) + \text{Error}(30)$

and $\text{Error}(30) \approx \dfrac{\text{Error}(10)}{3^2} \Rightarrow \text{Error}(10) \cong 9\,\text{Error}(30)$

This is a system with 3 unknowns and 3 equations
So we can solve for A, using the error!

$A = 12.676 + \text{Error}(10) = 12.676 + 9\,\text{Error}(30)$

$A = 10.420 + \text{Error}(30)$

so $12.676 + 9\,\text{Err}(30) = 10.420 + \text{Err}(30)$

$8\,\text{Error}(30) = -2.256$

$\text{Error}(30) = -.282$

so $A = \text{Trap}(30) + \text{Error}(30)$

$\approx 10.420 - .282$

$= 10.138$

So $\int_a^b f(x)dx \approx \underline{10.138}$