Math 265B: Test 3 In Class Name: KEY
(60 points)
For maximum credit, please show all of your work in a clear, logical manner. Simplify all answers as much as possible. Leave answers in exact form unless an approximation is asked for in the instructions.

1. (4 pts) Algebraically, determine whether the function $y = e^{-3t} + 2t - \frac{2}{3}$ is a solution to the Initial Value Problem $y' + 3y = 6t$, $y(0) = \frac{1}{3}$ Show work that justifies your answer.

$y = e^{-3t} + 2t - \frac{2}{3}$

$y' = -3e^{-3t} + 2$

$y' + 3y = 6t$

$(-3e^{-3t} + 2) + 3(e^{-3t} + 2t - \frac{2}{3}) \stackrel{?}{=} 6t$

$-3e^{-3t} + 2 + 3e^{-3t} + 6t - 2 \stackrel{?}{=} 6t$

$6t = 6t$ true

Also $y(0) = e^{-3(0)} + 2(0) - \frac{2}{3}$
$= 1 - \frac{2}{3} = \frac{1}{3}$

The function satisfies both the D.E. and the I.C.

Solution? Yes No

2. (4 pts) Find the solution to the following Initial Value Problem analytically.
$y' = .5(t-1)(y^2+1)$, $y(-1) = 0$

$\int \frac{dy}{y^2+1} = \int .5(t-1)\,dt$

$\tan^{-1}(y) = .5\left(\frac{1}{2}t^2 - t\right) + C$

$y = 0$ for $t = -1$

$\tan^{-1}(0) = .25(-1)^2 - .5(-1) + C$

$0 = .25 + .5 + C$

$C = -.75$

$\tan^{-1}(y) = .25t^2 - .5t - .75$

$\boxed{y = \tan(.25t^2 - .5t - .75)}$

3. (6 pts) Which of the following differential equations are separable? (Circle those that are.)

(i) $y' = t + y$ (ii) $y' = \cos(t)\tan(y)$ (iii) $y' = 4ty + 8y$ (iv) $y' = e^{t-y}$

Choose one of the equations above and solve it (find the general solution). Put the solution into explicit form.

(ii) $\frac{dy}{dt} = \cos(t)\tan(y)$

$\int \frac{dy}{\tan(y)} = \int \cos(t)\,dt$

$\int \frac{\cos(y)}{\sin(y)}\,dy = \sin(t) + C$

$\ln|\sin(y)| = \sin(t) + C$

$\sin(y) = e^{\sin(t) + C}$

$\sin(y) = Ce^{\sin(t)}$

$y = \sin^{-1}\left(Ce^{\sin(t)}\right)$

for (iii) and (iv), see attached

(iii) $y' = 4ty + 8y$

$$\frac{dy}{dt} = 4y(t+2)$$

$$\int \frac{dy}{y} = \int 4(t+2)\,dt$$

$$e^{\ln|y|} = e^{4\left(\frac{1}{2}t^2+2t\right)+C}$$

$$|y| = e^C e^{2t^2+8t}$$

$$\boxed{y = Ce^{2t^2+8t}}$$

(iv) $y' = e^{t-y}$

$$\frac{dy}{dt} = e^t e^{-y}$$

$$\int e^y\,dy = \int e^t\,dt$$

$$e^y = e^t + C$$

$$\ln(e^y) = \ln(e^t + C)$$

$$\boxed{y = \ln(e^t + C)}$$

4. (5 pts) Consider the Initial Value Problem $\frac{dy}{dt} = 3y(y-4)$, $y(0) = y_0$.

(a) Find any equilibrium solutions and classify them as stable or unstable.

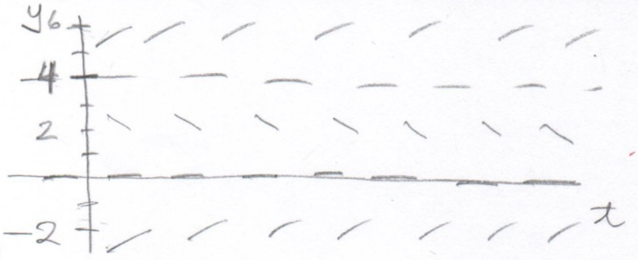
$\frac{dy}{dt} = 0$

$3y(y-4) = 0$

$\boxed{y=0}$ $\boxed{y=4}$

STABLE UNSTABLE

(b) Sketch the slope (direction) field for the differential equation.

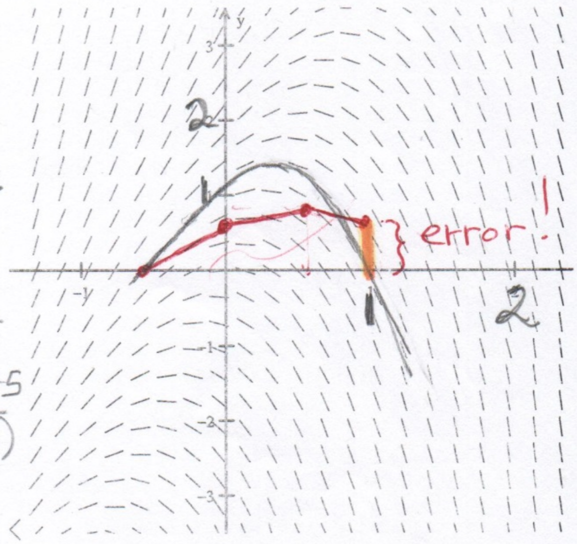


(c) For what initial conditions (i.e., what values of y_0) will the solution curves be increasing functions? $y_0 < 0$ and $y_0 > 4$

5. (6 pts) Consider the I.V.P. $\frac{dy}{dt} = y - 2t$, $y(-.5) = 0$

(a) Use Euler's Method, with $\Delta t = 0.5$ to estimate the value of $y(1)$. Clearly show all of your work.

n	(t_n, y_n)	$m = y_n - 2t_n$	$y_{n+1} = y_n + m\Delta t$
0	$(-.5, 0)$	$m = 0 - 2(-.5) = 1$	$y_1 = y_0 + 1(.5) = 0 + .5 = .5$
1	$(0, .5)$	$m = .5 - 2(0) = .5$	$y_2 = y_1 + .5(.5) = .5 + .25 = .75$
2	$(.5, .75)$	$m = .75 - 2(.5) = -.25$	$y_3 = y_2 + (-.25)(.5) = .75 - .125 = .625$
3	$(1, .625)$		



$y(1.0) \cong$ .625

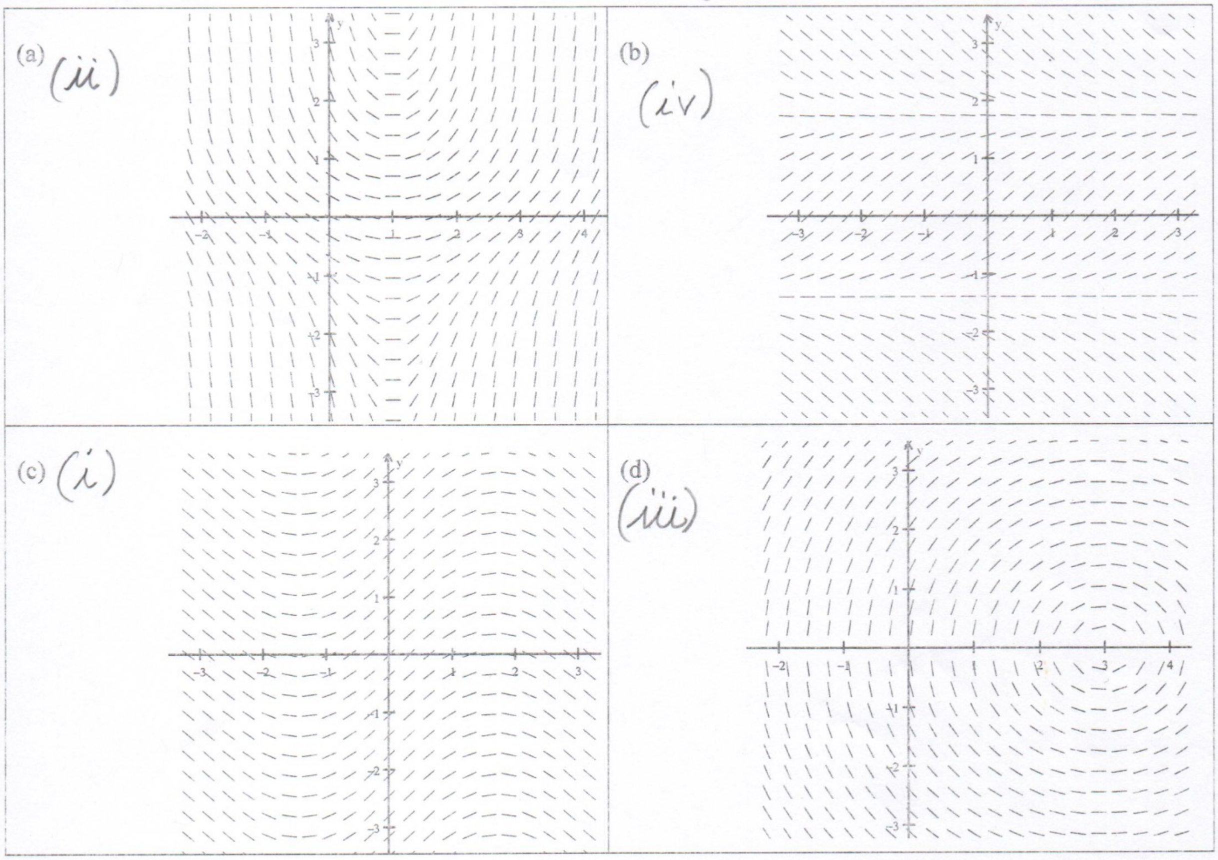
(b) On the slope field provided, sketch the polygonal Euler's solution "curve", based on your work from part (a) *and also sketch* the solution curve that passes through the initial condition, based on just the slope field.

(c) Based on your work above, is the Euler's Method value for y(1) a good approximation for the actual solution value? Why or why not? Give an approximation for the error, based just on your graphs.

The rough sketch of the solution curve puts the value of $y(1)$ around 0 ($y(1) \approx 0$), so the Euler's Method approximation is not a very good estimate. Error looks like around .5 or .6.

6. (4 pts) Label the slope fields with the one differential equation that best matches it. No work need be shown.

(i) $y' = -\cos(t+3)$ (c) (ii) $y' = .5(t-1)(y^2+1)$ (a) (iii) $y' = \dfrac{3-t}{y}$ (d) (iv) $y' = -\cos(y+3)$ (b)



7. (3 pts) Use the Squeeze Theorem to prove that $\lim\limits_{n\to\infty} e^{-n}\sin(n) = 0$

We know that $-1 \le \sin(n) \le 1$

~~so~~ $-\dfrac{1}{e^n} \le \dfrac{\sin(n)}{e^n} \le \dfrac{1}{e^n}$

but $\lim\limits_{n\to\infty}\left(-\dfrac{1}{e^n}\right) = 0$ and $\lim\limits_{n\to\infty}\left(\dfrac{1}{e^n}\right) = 0$

$\therefore$ by the Squeeze Theorem, we know that

$\lim\limits_{n\to\infty} e^{-n}\sin(n) = \lim\limits_{n\to\infty} \dfrac{\sin(n)}{e} = 0$ Q.E.D.

8. (8 pts) Using analytic techniques, i.e. *formally*, find the limit of each the following sequences (as n approaches infinity). If you are using a theorem in your analysis of the limit, write the name of the theorem. *If the limit does not exist, explain why it doesn't.*

(a) $a_n = \dfrac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{n^n}{n!} = \boxed{\infty}$$

by the Ranking Theorem

(b) $a_n = \dfrac{\ln(n^2)}{\ln(n)}$

$$\lim_{n\to\infty} \frac{\ln(n^2)}{\ln(n)}$$

$$= \lim_{n\to\infty} \frac{2\cancel{\ln(n)}}{\cancel{\ln(n)}}$$

$$= \lim_{n\to\infty} 2 = \boxed{2}$$

(You can also apply L.H. Rule)

$a_n = \dfrac{3n}{2n+1}$

$$\lim_{n\to\infty} \frac{3n}{2n+1} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{3}{2} \quad \text{By L'H. Rule}$$

$$= \boxed{\frac{3}{2}}$$

(d) $a_n = \cos(\pi n)$

$\lim_{n\to\infty} \cos(\pi n)$ dne

since the sequence just oscillates between -1 and 1 and never approaches a single number.

For rest of the exam, you may find any limits "by inspection".

9. (3 pts) Given the series $\sum_{k=0}^{\infty} \frac{2}{3}(-1)^k$

(a) Find the first 4 partial sums of the series.

$S_1 = \frac{2}{3}(-1) = -\frac{2}{3}$

$S_2 = \frac{2}{3}(-1) + \frac{2}{3} = 0$

$S_3 = \frac{2}{3}(-1) + \frac{2}{3} + \frac{2}{3}(-1) = -\frac{2}{3}$

$S_4 = -\frac{2}{3} + \frac{2}{3} - \frac{2}{3} + \frac{2}{3} = 0$

(b) Does this series converge? No, $\lim_{n\to\infty} S_n$ dne

10. (4 pts) Given the series $\sum_{k=1}^{\infty} \frac{3}{k} - \frac{3}{k+1}$

(a) Find the first 4 partials sums of the series. Show work!

$$S_1 = \frac{3}{1} - \frac{3}{2} = 1.5$$

$$S_2 = \left(3 - \cancel{\frac{3}{2}}\right) + \left(\cancel{\frac{3}{2}} - \frac{3}{3}\right) = 2$$

$$S_3 = \left(3 - \cancel{\frac{3}{2}}\right) + \left(\cancel{\frac{3}{2}} - \cancel{\frac{3}{3}}\right) + \left(\cancel{\frac{3}{3}} - \frac{3}{4}\right) = 2.25$$

$$S_4 = \left(3 - \cancel{\frac{3}{2}}\right) + \left(\cancel{\frac{3}{2}} - \cancel{\frac{3}{3}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{3}{4}} - \frac{3}{5}\right) = 2.4$$

(b) Write the nth term of the sequence of partial sums.

$$S_n = 3 - \frac{3}{n+1}$$

could be written as $S_n = \frac{3n}{n+1}$

(c) Use your work above to determine whether the series converges and if so, the value it converges to.

Since $\lim_{n \to \infty} S_n = \lim_{n \to \infty} 3 - \frac{3}{n+1} = 3$

the series converges, and converges to 3

$$\boxed{\sum_{k=1}^{\infty} \frac{3}{k} - \frac{3}{k+1} = 3}$$

11. (3 pts) For the sequence $1, -2, 4, -8, \ldots$

(a) find the next two terms 16, -32

(b) find a recurrence relation for the sequence

$$\boxed{a_{n+1} = -2a_n, \quad a_1 = 1}$$

12. (3 pts) For the sequence $\frac{1}{3}, \frac{4}{9}, \frac{7}{27}, \frac{10}{81}, \ldots$

(a) find the next two terms $\frac{13}{243}$, $\frac{16}{729}$

(b) Find an explicit formula (nth term formula) for the sequence .

$$a_n = \frac{3n-2}{3^n}, \quad n = 1, 2, 3, \cdots$$

13. (7 pts) Determine which of the following geometric series converges. For those that converge, find the sum of the series.

(a) $\sum_{k=0}^{\infty}\left(-\frac{4}{3}\right)^k$ diverges

$$|r| = \left|-\frac{4}{3}\right| = \frac{4}{3} > 1$$

$\sum_{k=0}^{\infty}\frac{2^k}{3^{k+2}} = \sum_{k=0}^{\infty} \frac{1}{3^2} \cdot \frac{2^k}{3^k}$ Converges

$$= \sum_{k=0}^{\infty} \frac{1}{9}\left(\frac{2}{3}\right)^k$$

$$= \frac{\frac{1}{9}}{1-\frac{2}{3}} = \boxed{\frac{1}{3}}$$

(b) $\sum_{k=0}^{\infty}100(.8)^k$ converges

$$= \frac{100}{1-.8} = \frac{100}{.2} = \boxed{500}$$

Converges

$\sum_{k=1}^{\infty}\frac{1}{5^k} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$

$$= \frac{\frac{1}{5}}{1-\frac{1}{5}}$$

$$= \frac{\frac{1}{5}}{\frac{4}{5}} = \boxed{\frac{1}{4}}$$