Math 265B: Test 3 Name: KEY

For maximum credit, please show all of your work in a clear, logical manner. Simplify all answers as much as possible. Leave answers in exact form unless an approximation is asked for in the instructions.

1. (3 pts) True or false:

(a) Euler's Method gives the exact solution to an initial value problem. TRUE **FALSE**

(b) The parameterization of a curve is always unique; i.e., there is only one set of parametric equations for any curve. TRUE **FALSE**

(c) The polar coordinates $(2,0)$ and $(-2,\pi)$ describe the same point. **TRUE** FALSE

2. (6 pts) CHECK (don't attempt to solve!) to see whether $y = t + \frac{1}{t}$ is a solution to the differential equation $ty' + y = 2t$. Be sure to state your conclusion (the function <u>is</u> or <u>is not</u> a solution to the DE). <u>*Show work that justifies your answer*</u>.

$y = t + \frac{1}{t}$

$y' = 1 - \frac{1}{t^2}$

$ty' + y = 2t$

$t\left(1 - \frac{1}{t^2}\right) + \left(t + \frac{1}{t}\right) \stackrel{?}{=} 2t$

$t - \frac{1}{t} + t + \frac{1}{t} \stackrel{?}{=} 2t$

$2t = 2t$

true $\therefore$ $y = t + \frac{1}{t}$ IS a solution.

3. (4 pts) Which of the following differential equations are separable? (Circle those that are.)

(i) $y' = t^2y^2$ (ii) $y' = 3t + y$ (iii) $y' = (y-1)(y+3)$ (iv) $y' = \sin(ty)$

4. (8 pts) Find the general solution (in explicit form) for each differential equation. Carefully show all steps when simplifying your answer.

(a) $y' = \frac{3+t^2}{\sqrt{y}}$ $\Rightarrow$ $\frac{dy}{dt} = \frac{3+t^2}{\sqrt{y}}$

$\int \sqrt{y}\, dy = \int 3 + t^2\, dt$

$\frac{2}{3}y^{3/2} = 3t + \frac{1}{3}t^3 + C$

$y^{3/2} = \frac{9}{2}t + \frac{1}{2}t^3 + C$

$\boxed{y = \left(\frac{9}{2}t + \frac{1}{2}t^3 + C\right)^{2/3}}$

(b) $y' + 5y = 10$

$\frac{dy}{dt} = -5y + 10$

$\frac{dy}{dt} = -5(y-2)$

$\int \frac{dy}{y-2} = \int -5\, dt$

$e^{\ln|y-2|} = e^{-5t + C}$

$y - 2 = \pm e^C e^{-5t}$ $C_1 = \pm e^C$

$\boxed{y = 2 + C_1 e^{-5t}}$

5. (8 pts) Solve the Initial Value Problem.

$\frac{dy}{dt} = e^{-y}\sin(t), \quad y(\frac{\pi}{2}) = 0$ $\Rightarrow y=0$ when $t=\frac{\pi}{2}$

$\int e^{y}\,dy = \int \sin t\,dt$

$e^{y} = -\cos t + C$

evaluate C:

$t=\frac{\pi}{2},\ y=0$

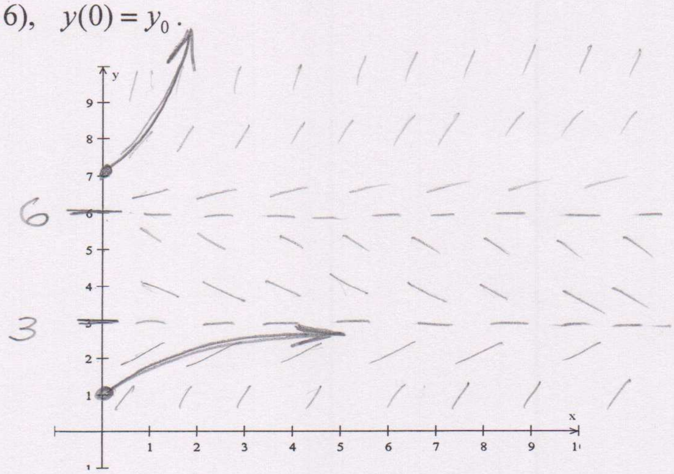
$e^{0} = -\underbrace{\cos\frac{\pi}{2}}_{0} + C \Rightarrow C=1$

$e^{y} = -\cos t + 1$

$\boxed{y = \ln(-\cos t + 1)}$

6. (8 pts) Consider the Initial Value Problem $\frac{dy}{dx} = (y-3)(y-6), \quad y(0) = y_0$.

(a) Sketch the slope field for the differential equation.



(b) Find any equilibrium (constant) solutions and classify each as stable or unstable.

$y=3$ stable

$y=6$ unstable

(c) Sketch solution curves for the Initial Conditions $y(0)=1$ and $y(0)=7$

(d) For what initial conditions (i.e., what values of y_0) will the solution curves be decreasing functions?

For $3 < y_0 < 6$

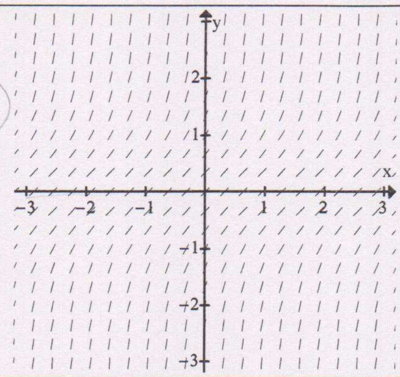
7. (8 pts) Label each slope field with the differential equation that best matches it. No work is required to be shown.

(i) $y'(t) = \dfrac{t}{2+y}$ (ii) $y'(t) = \cos(t+y)$ (iii) $y'(t) = 1 + y^2$ (iv) $y'(t) = ty$

(a) (iii)

$y' = 1 + y^2$

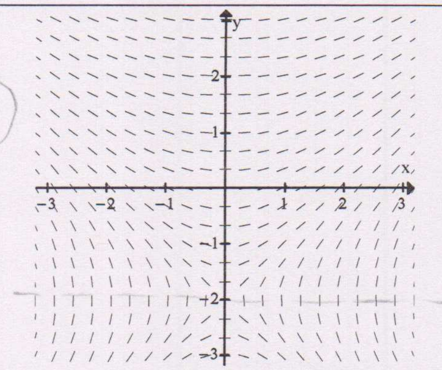
$y = 0$
$y' = 1$



(b) (i)

$y' = \dfrac{t}{2+y}$

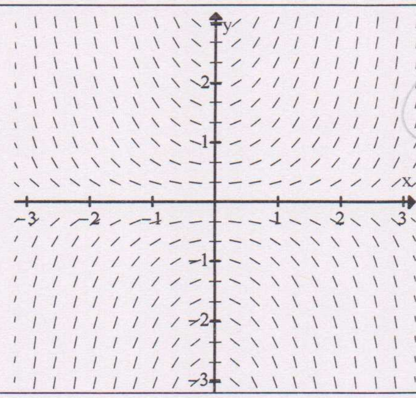
$y = -2$
vertical tangent



(c) (iv)

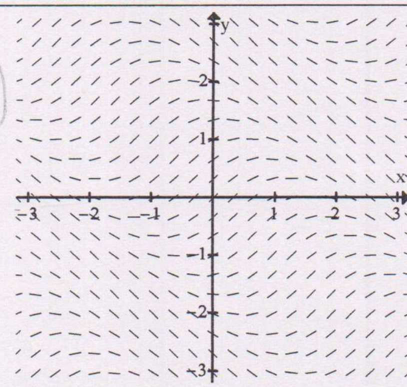
$y' = ty$

at $(0,0)$
$m = 0$



(d) ii

$y' = \cos(t+y)$

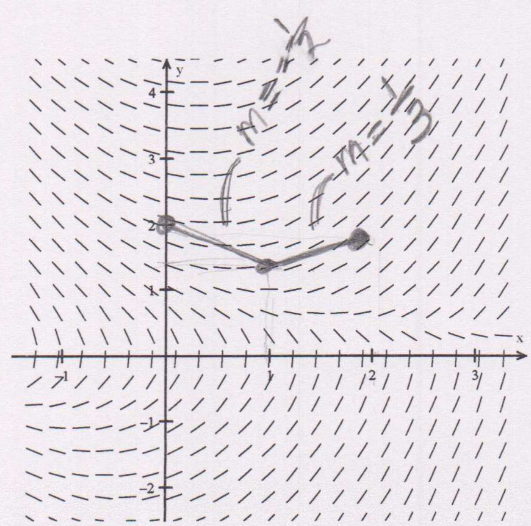


8. (10 pts) Consider the I.V.P. $y' = t - \frac{1}{y}$ $y(0) = 2$

(a) Use Euler's Method, with $\Delta t = 1$ to estimate the value of $y(2)$. Clearly show all of your work.

n	t_n	y_n	$m = t_n - \frac{1}{y_n}$	$y_{n+1} = y_n + m\Delta t$
0	0	2	$m = 0 - \frac{1}{2} = -\frac{1}{2}$	$y_1 = y_0 + (-\frac{1}{2})(1) = 2 - \frac{1}{2} = \frac{3}{2}$
1	1	$\frac{3}{2}$	$m = 1 - \frac{2}{3} = \frac{1}{3}$	$y_2 = y_1 + (\frac{1}{3})(1) = \frac{3}{2} + \frac{1}{3} = \frac{11}{6} = 1\frac{5}{6}$
2	2	$\frac{11}{6}$		

$y(2) \cong$ $\underline{\frac{11}{6} = 1\frac{5}{6}}$



(b) On the slope field provided, sketch the polygonal Euler's solution "curve", based on your work from part (a).

9. (4 pts) A cup of coffee cools at a rate proportional to the difference between the coffee's temperature, H, and the ambient temperature, A.

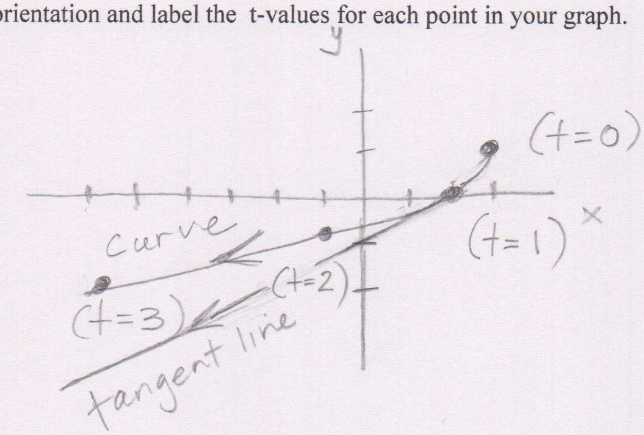
Set up a differential equation that models this relationship. Do not solve the D.E.

$$\boxed{\frac{dH}{dt} = k(H - A)}$$

10. (12 pts) Consider the curve described by the parameterization $\begin{cases} x = 3 - t^2 \\ y = 1 - t \end{cases}$ $0 \le t \le 3$

(a) Graph the curve. Include the orientation and label the t-values for each point in your graph.

t	x	y
0	3	1
1	2	0
2	-1	-1
3	-6	-2



(b) Find the *parametric form* of the tangent line equation at the point where t = 1. Sketch the line on your graph from part (a).

$$\left.\frac{dx}{dt}\right|_{t=1} = -2t \Big|_{t=1} = -2$$

$$\left.\frac{dy}{dt}\right|_{t=1} = -1$$

$$(x_0, y_0) = (x(1), y(1)) = (2, 0)$$

Tangent line:

$$\begin{cases} x = 2 - 2(t-1) \\ y = -(t-1) \end{cases}$$
$$t \ge 1$$

(c) Eliminate the parameter and express the curve as an xy-equation.

$y = 1 - t$

$t = 1 - y$ $\rightarrow$ $x = 3 - t^2$

$$\boxed{x = 3 - (1-y)^2}$$

11. (12 pts) Find a parameterization of each of the following curves. Be sure to include the domain!

2 (a) The segment of the parabola $y = x^2 + 3$ where $-1 \le x \le 3$

$$\boxed{\begin{cases} x = t \\ y = t^2 + 3 \end{cases} \quad -1 \le t \le 3}$$

4 (b) The line segment that begins at point P(4, -5) and ends and point Q(1,-1)

$$\boxed{\begin{cases} x = 4 - 3t \\ y = -5 + 4t \end{cases} \quad 0 \le t \le 1}$$

4 (c) A circle of radius 3, oriented clockwise, with center (3, 2) and period of 10 seconds.

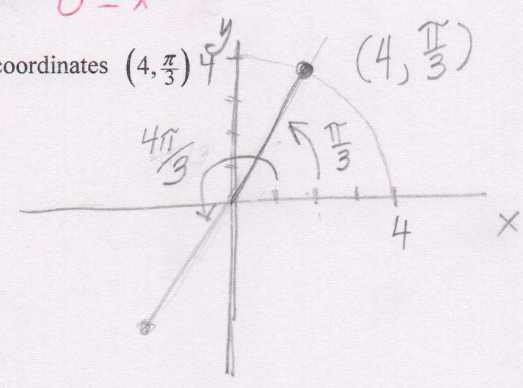
$$\begin{cases} x = 3\cos\left(\frac{\pi}{5}t\right) + 3 \\ y = -3\sin\left(\frac{\pi}{5}t\right) + 2 \end{cases} \quad 0 \le t \le 10 \qquad B = \frac{2\pi}{10} = \frac{\pi}{5}$$

2 (d) The top half of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, oriented counter clockwise, starting at point (2, 0)

$$\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$$

$0 \le t \le \pi$ $\quad 0 \le t \le \pi$

12. (6 pts) (a) Plot the point with polar coordinates $\left(4, \frac{\pi}{3}\right)$



(b) Find two other sets of polar coordinates that locate that same point. One set must have a negative r-value.

$\boxed{\left(-4, \frac{4\pi}{3}\right)}$ also $\left(-4, -\frac{2\pi}{3}\right)$

$\boxed{\left(4, \frac{7\pi}{3}\right)}$ also $\left(4, -\frac{5\pi}{3}\right)$

(c) Convert the point into rectangular coordinates. For full credit, express your answer in EXACT terms.

$x = 4\cos\frac{\pi}{3} = 4\left(\frac{1}{2}\right) = 2$

$y = 4\sin\frac{\pi}{3} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$

$\boxed{(2, 2\sqrt{3})}$

13. (6 pts) Convert each of the xy-equations into a polar equation.

(a) $x^2 + y^2 = 7$

$r^2 = 7$

OR

$r = \sqrt{7}$

(b) $y = -6$

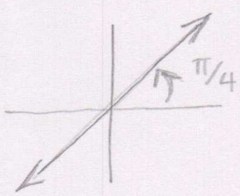
$r\sin\theta = -6$

OR

$r = \dfrac{-6}{\sin\theta}$

(c) $y = x$ (express your final answer in terms of θ only).

$\theta = \dfrac{\pi}{4}$



OR derive it by

$r\sin\theta = r\cos\theta$

$\dfrac{\sin\theta}{\cos\theta} = 1$

$\tan\theta = 1$

$\theta = \tan^{-1}(1)$

$\theta = \dfrac{\pi}{4}$

14.(4 pts) Graph the region $\left\{(r,\theta) \middle| \; 2 \le r \le 3, \; -\frac{\pi}{6} \le \theta \le \frac{\pi}{6}\right\}$