Math 265B: Test 4 - In Class Portion

Name: KEY

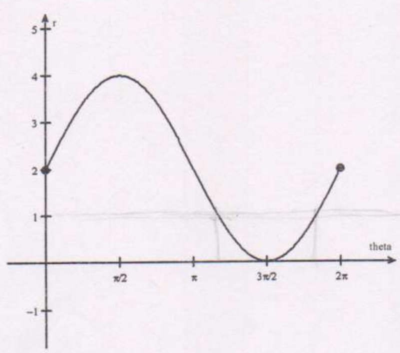
(76 pts)

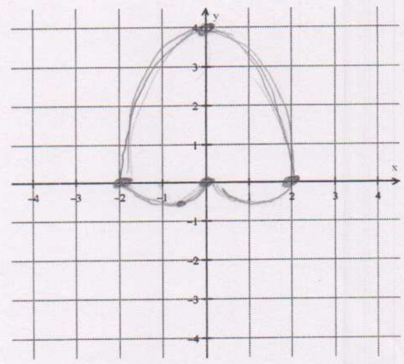
For maximum credit, please show all of your work in a clear, logical manner. Simplify all answers as much as possible. Leave answers in exact form unless an approximation is asked for in the instructions.

Helpful formula: $\dfrac{dy}{dx} = \dfrac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$

1.(4 pts) Use the Cartesian r, θ graph to graph the polar equation in the xy-coordinate system.

$r = 2 + 2\sin\theta$

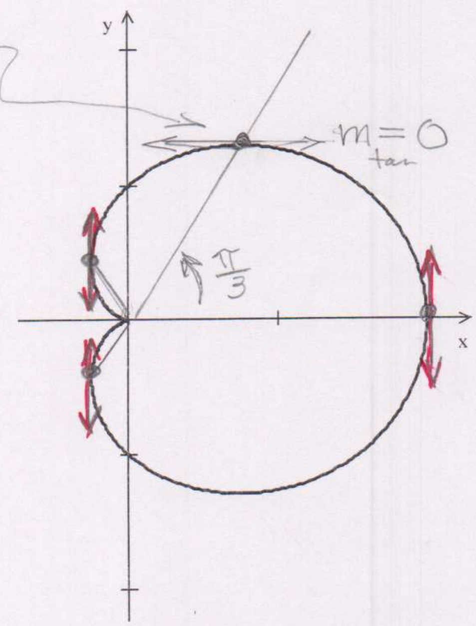




2. (6 pts) Consider the polar curve $r = 1 + \cos\theta$.

Use calculus to determine the slope of the curve at the point $\left(\frac{3}{2}, \frac{\pi}{3}\right)$.

Plot the point on the curve, sketch the tangent line, and label it with the slope.



$$\frac{dy}{dx} = \frac{-\sin\theta\sin\theta + (1+\cos\theta)\cos\theta}{-\sin\theta\cos\theta - (1+\cos\theta)\sin\theta}$$

$$= \frac{-\sin^2\theta + \cos\theta + \cos^2\theta}{-2\sin\theta\cos\theta - \sin\theta}$$

(opt.) $$= \frac{\cos(2\theta) + \cos\theta}{-\sin(2\theta) - \sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{3}} = \frac{\cos(2\pi/3) + \cos(\pi/3)}{-\sin(2\pi/3) - \sin(\pi/3)}$$

$$= \frac{-\frac{1}{2} + \frac{1}{2}}{-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}} = 0$$

$\boxed{m_{tan} = 0 \text{ at } \left(\frac{3}{2}, \frac{\pi}{3}\right)}$

Extra credit (4 pts) (do this only if you've finished the entire exam!):

Use the graph to estimate the values of θ at which the curve has a vertical tangent line. Verify your guess analytically. i.e., algebraically solve for your original guess. (Do this calculation on the next page.)

By inspection, tangent is vertical at $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

Extra credit work (optional):

To find where tangent is vertical, set denom. = 0 and solve for θ:

$-\sin 2\theta - \sin\theta = 0$

$-2\sin\theta\cos\theta - \sin\theta = 0$

$-\sin\theta\,(2\cos\theta + 1) = 0 \Rightarrow 2\cos\theta + 1 = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$-\sin\theta = 0$ at $\theta = 0, \pi, 2\pi$, etc

$\theta = \pi$, slope is undefined (cusp)

$\Rightarrow \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

which confirms the original guess.

3. (8 pts) (a) Find the points of intersection for the polar curves $r = \frac{1}{\sqrt{2}}$ and $r = \sqrt{\sin\theta}$. The graph of the curves is given.

Solve $\begin{cases} r = \frac{1}{\sqrt{2}} \\ r = \sqrt{\sin\theta} \end{cases}$

$\left(\frac{1}{\sqrt{2}}\right)^2 = \left(\sqrt{\sin\theta}\right)^2$

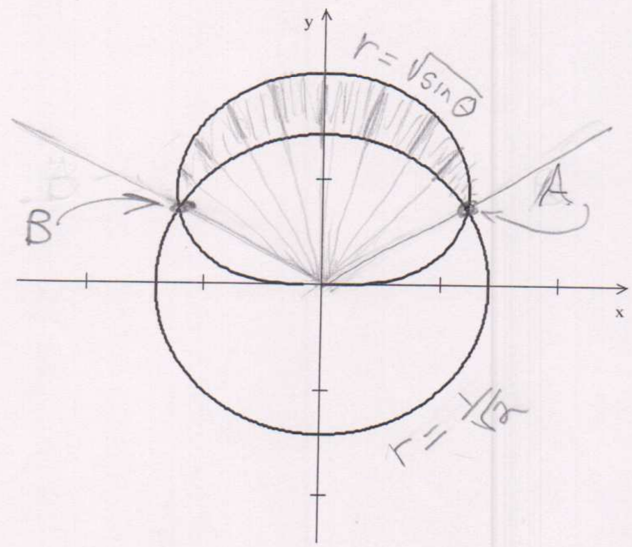
$\frac{1}{2} = \sin\theta$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

points:

A: $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right)$

B: $\left(\frac{1}{\sqrt{2}}, \frac{5\pi}{6}\right)$



Find the area of the region, that is outside of the curve $r = \frac{1}{\sqrt{2}}$ and inside the curve $r = \sqrt{\sin\theta}$.

$$A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}\left(\sqrt{\sin\theta}\right)^2 - \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 d\theta$$

using symmetry

$$= \int_{\pi/6}^{\pi/2} \sin\theta - \frac{1}{2}\, d\theta$$

$$= -\cos\theta\Big|_{\pi/6}^{\pi/2} - \frac{1}{2}\theta\Big|_{\pi/6}^{\pi/2} = 0 - \left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

Total area is $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$

Extra credit work (optional):

To find vertical tangent set denom = 0 $\Longrightarrow \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$ by inspection and confirmed algebraically.

$-\sin(2\theta) - \sin\theta = 0$

$-2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(-2\cos\theta - 1) = 0$

$\sin\theta = 0$ $\quad\boxed{\theta = 0}$

$-2\cos\theta = 1$

$\cos\theta = -\frac{1}{2}$

$\boxed{\theta = 2\pi/3}$ (QII) and $\boxed{\theta = \frac{4\pi}{3}}$ (QIII)

3. (8 pts) (a) Find the points of intersection for the polar curves $r = \frac{1}{\sqrt{2}}$ and $r = \sqrt{\sin\theta}$. The graph of the curves is given.

3 pts

$r = \frac{1}{\sqrt{2}}$

$r = \sqrt{\sin\theta}$

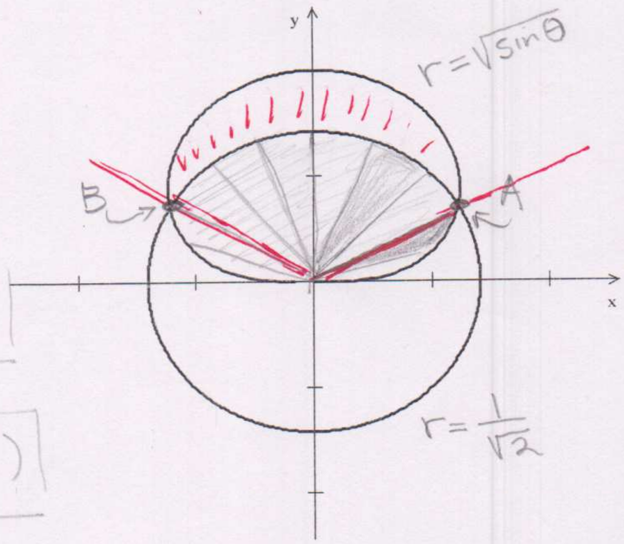
$\Longrightarrow \frac{1}{\sqrt{2}} = \sqrt{\sin\theta}$

$\frac{1}{2} = \sin\theta$

$\Longrightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

point A: $\theta = \frac{\pi}{6}, r = \frac{1}{\sqrt{2}}$ $\boxed{\left(\frac{1}{\sqrt{2}}, \frac{\pi}{6}\right)}$

point B: $\theta = \frac{5\pi}{6}, r = \frac{1}{\sqrt{2}}$ $\boxed{\left(\frac{1}{\sqrt{2}}, \frac{5\pi}{6}\right)}$



(b) Find the area of the region, that is outside of the curve $r = \frac{1}{\sqrt{2}}$ and inside the curve $r = \sqrt{\sin\theta}$.

5 pts Find Area from $\theta = 0$ to $\frac{\pi}{2}$ then double it...

$$A_{\frac{1}{2}} = \int_0^{\pi/6} \frac{1}{2}\left(\sqrt{\sin\theta}\right)^2 d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 d\theta$$

$$= \int_0^{\pi/6} \sin\theta \, d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{4} d\theta$$

$$= -\cos\theta \Big|_0^{\pi/6} + \frac{1}{4}\theta \Big|_{\pi/6}^{\pi/2}$$

$$= -\frac{\sqrt{3}}{2} + 1 + \frac{1}{4}\left(\frac{\pi}{2}\right) - \frac{1}{4}\left(\frac{\pi}{6}\right)$$

$\Longrightarrow A_{\frac{1}{2}} = 1 - \frac{\sqrt{3}}{2} + \frac{\pi}{12}$

Total Area:

$\boxed{A = 2A_{\frac{1}{2}} = 2 - \sqrt{3} + \frac{\pi}{6}}$

4. (14 pts) Using analytic techniques, i.e. *formally*, find the following limits. The methods/theorems you may use include the Ranking Theorem, L'Hopital's Rule, algebraic manipulation, and/or the Squeeze Theorem. *If the limit does not exist, briefly explain why.* Caution: THESE ARE NOT SERIES! DO NOT USE SERIES CONVERGENCE TESTS!

3 (a) $\lim_{n\to\infty} \frac{n^{50}}{2^n} = 0$ by the Ranking Theorem

3 (b) $\lim_{n\to\infty} \frac{n^2 - n}{4n^2+1} = \lim_{n\to\infty} \frac{2n-1}{8n} = \lim_{n\to\infty} \frac{2}{8} = \frac{1}{4}$ by L'Hôpital's Rule

4 (c) $\lim_{n\to\infty} \frac{\tan^{-1}(n)}{n} = 0$ by the Squeeze Theorem

$-\frac{\pi}{2} < \tan^{-1}(n) < \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2n} < \frac{\tan^{-1}(n)}{n} < \frac{\pi}{2n}$

Since $\lim_{n\to\infty} \left(-\frac{\pi}{2n}\right) = 0$

and $\lim_{n\to\infty} \left(\frac{\pi}{2n}\right) = 0$, we must have

4 (d) $\lim_{n\to\infty} \cos(\pi n)$

$= \lim_{n\to\infty} (-1)^n$ dne

Since the sequence value never settles down to be approaching a single number, but instead flip flops between 1 ad -1 endlessly.

For rest of the exam, you may find any limits "by inspection".

5. (8 pts) Given the series $\sum_{k=0}^{\infty} \frac{3}{k+1} - \frac{3}{k+2}$

3 (a) Find the first 3 partial sums of the series.

$$S_1 = \underbrace{\frac{3}{1} - \frac{3}{2}}_{k=0} = 1.5$$

$$S_2 = \underbrace{\frac{3}{1} - \cancel{\frac{3}{2}}}_{k=0} + \underbrace{\cancel{\frac{3}{2}} - \frac{3}{3}}_{k=1} = 2$$

$$S_3 = \underbrace{\frac{3}{1} - \cancel{\frac{3}{2}}}_{k=0} + \underbrace{\cancel{\frac{3}{2}} - \cancel{\frac{3}{3}}}_{k=1} + \underbrace{\cancel{\frac{3}{3}} - \frac{3}{4}}_{k=2} = 2.25$$

2 (b) What is the ***nth*** partial sum of the series?

$$S_n = 3 - \frac{3}{n+1}$$

3 (c) Use your work above to show that the series converges and to find the sum of the series. Show careful work that justifies your answer. ☆

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 3 - \frac{3}{n+1} = 3 \text{☆}$$

∴ since the limit of the partial sums exists, the series converges to 3.
The sum of the series is 3.

6. (6 pts) Write each of the following series in $\sum$ notation, <u>and</u> determine whether the series converges or diverges. You do not have to prove convergence/divergence, just determine "by inspection".

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots =$ $\sum_{K=1}^{\infty} (-1)^{K+1} \frac{1}{K}$ converges diverges

(b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots =$ $\sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^K$ converges diverges

(c) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \cdots =$ $\sum_{K=1}^{\infty} \frac{K}{K+1}$ converges diverges

(answers may vary)

7. (6 pts) If the series below converges, find its sum. Show work. If it doesn't converge, say how you can tell.

$$\sum_{k=0}^{\infty} 2\left(\tfrac{3}{4}\right)^k + 4\left(-\tfrac{1}{3}\right)^k$$

These are just geometric series, both with $|r|<1$, so they both converge...

$$= 2\sum_{k=0}^{\infty}\left(\tfrac{3}{4}\right)^k + 4\sum_{k=0}^{\infty}\left(-\tfrac{1}{3}\right)^k$$

By the properties of convergent series... we can break the series apart... and thus get the sum!

$$= 2\cdot\left(\frac{1}{1-\frac{3}{4}}\right) + 4\cdot\left(\frac{1}{1+\frac{1}{3}}\right)$$

$$= 2\cdot 4 + 4\cdot\tfrac{3}{4} = \boxed{11}$$

For each of the following problems, 8 - 11, determine convergence and organize your work in the "Claim…Work…Conclusion" format.

8. (5 pts) Consider the series $\sum_{k=0}^{\infty}\frac{1}{k^2+1}$. Use Direct Comparison to prove that the series converges or diverges.

Claim: Series converges

Work: since $k^2 < k^2+1$

we have $\frac{1}{k^2+1} < \frac{1}{k^2}$

but $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by the p-Test so

conclusion: $\sum_{k=1}^{\infty}\frac{1}{k^2+1}$ converges by Direct Comparison.

9. (5 pts) Use the Ratio Test to prove that the series converges or diverges.

$$\sum_{k=1}^{\infty}\frac{k^3}{3^k}$$

Claim: Series converges

Work: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$

$$= \lim_{k\to\infty}\left|\frac{\frac{(k+1)^3}{3^{k+1}}}{\frac{k^3}{3^k}}\right|$$

$$= \lim_{k\to\infty}\frac{(k+1)^3}{3k^3} = \frac{1}{3} < 1$$

Conclusion: by the Ratio Test the series converges.

10. (8 pts) Use any test to prove convergence or divergence of the series.

(a) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{k^3+1}{k^3+2k}$

Claim: Series diverges

Work: $\lim_{k\to\infty} (-1)^k \frac{k^3+1}{k^3+2k}$ dne

since the terms oscillate without damping

Conclusion: Since $\lim_{k\to\infty} a_k \neq 0$ the series diverges by the Divergence Test.

(b) $\displaystyle\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$

Claim: Series converges

Work: $\lim_{k\to\infty} \dfrac{\frac{(k+1)!(k+1)!}{(2(k+1))!}}{\frac{k!\,k!}{(2k)!}}$

$= \lim_{k\to\infty} \frac{(k+1)k!(k+1)k!}{(2k+2)(2k+1)(2k)!} \cdot \frac{(2k)!}{k!k!}$

$= \lim_{k\to\infty} \frac{(k+1)(k+1)}{(2k+2)(2k+1)} = \frac{1}{4} < 1$

Conclusion: The series converges by the Ratio Test.

11. (6 pts) Determine whether the series is ***absolutely convergent***, ***conditionally convergent***, or ***divergent***.

$\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^{2/5}} = \sum (-1)^{k+1} a_k$

Is the series absolutely convergent? No

Show work to justify your answer:

$\displaystyle\sum_{k=1}^{\infty} \left|(-1)^{k+1} \frac{1}{k^{2/5}}\right| = \sum |a_k|$

$= \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{2/5}}$

which diverges by the p-Test

Is the series conditionally convergent? Yes

Show work to justify your answer:

① a_k decreases monotonically as $k \to \infty$ (by inspection)

② $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k^{2/5}} = 0$

$\therefore$ by the Alternating Series Test the series converges.