Math 265B: Test 4 Name: KEY

For maximum credit, please show all of your work in a clear, logical manner. Simplify all answers as much as possible. (Formulas, etc., are on the last page.)

1. (12 pts) Using analytic techniques, i.e. *formally*, find the limit of each the following sequences (as n approaches infinity). If you are using a theorem in your analysis of the limit, write the name of the theorem. If the limit does not exist, say why.

(a) $a_n = \dfrac{n^{100}}{n!}$

$\lim_{n\to\infty} \dfrac{n^{100}}{n!} = 0$ by the Ranking Theorem: $n^p \ll n!$

(b) $a_n = \sin\left(\frac{\pi n}{4}\right)$

$\lim_{n\to\infty} \sin\left(\frac{\pi n}{4}\right)$ dne due to oscillation without damping, thus the sequence values never settle to one single value

n	1	2	3	4	5	6	7	8
a_n	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0

(c) $a_n = \dfrac{\cos(n)}{n}$

$\lim_{n\to\infty} \dfrac{\cos(n)}{n} = 0$

Since $-1 \le \cos(n) \le 1$

we have $-\frac{1}{n} \le \frac{\cos(n)}{n} \le \frac{1}{n}$

Now $\lim_{n\to\infty} \left(-\frac{1}{n}\right) = 0$

and $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$

Thus by the Squeeze Theorem we must have that $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$

(d) $a_n = \dfrac{n^2+5n}{4n^2+1}$

Approach 1: Apply L'Hôpital's Rule

$$\lim_{n\to\infty} \frac{n^2+5n}{4n^2+1} = \lim_{n\to\infty} \frac{2n+5}{8n} = \lim_{n\to\infty} \frac{2}{8} = \frac{1}{4}$$

Approach 2: Algebraic Approach

$$\lim_{n\to\infty} \frac{(n^2+5n)(1/n^2)}{(4n^2+1)(1/n^2)} = \lim_{n\to\infty} \frac{1+\frac{5}{n}}{4+\frac{1}{n^2}} = \frac{1+0}{4+0} = \frac{1}{4}$$

For rest of the exam, you may find any limits "by inspection".

2. (4 pts) True or false (circle the correct answer):

(a) If $\sum_{k=0}^{\infty} a_k$ diverges very slowly then $\sum_{k=1000}^{\infty} a_k$ converges. No! True False

(b) If $\lim_{k\to\infty} b_k = \frac{1}{2}$ for the series $\sum_{k=0}^{\infty} b_k$ then the series diverges. True False

(c) Given two series, $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, if $\sum_{k=0}^{\infty} b_k$ converges

and $a_k \geq b_k$ and the limit of a_k is 0, then we know $\sum_{k=0}^{\infty} a_k$ converges True False

(d) The formula for Taylor polynomials, expanded about x = 0, was created by making a function value and its derivatives' values match the polynomial's value and its derivatives' values at x = 0. True False

3. (6 pts) Fill in and/ or circle the correct answer:

(a) $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges / diverges (circle one) because p-Test, $p = 3/2 > 1$

(b) $\sum_{k=0}^{\infty} \frac{1}{3}\left(\frac{\pi}{e}\right)^k$ converges / diverges (circle one) because Geometric Series, $r = \frac{\pi}{e} > 1$

(c) If the series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges conditionally, then the series $\sum_{k=1}^{\infty} |a_k|$ converges / diverges (circle one)

(d) The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the Harmonic Series and it converges / diverges (circle one).

(e) Does the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/3}}$ converge absolutely? No How do you know? p-Test, $p = \frac{1}{3} \leq 1$

Does it converge conditionally? Yes How do you know? $\lim_{k\to\infty} \frac{1}{k^{1/3}} = 0$ so by A.S.T. the alternating series converges.

4. (4 pts) Given the series $\sum_{k=0}^{\infty} 3(-1)^k$

(a) Find the first 4 partial sums of the series.

$S_0 = 3(-1)^0 = \boxed{3}$

$S_1 = 3(-1)^0 + 3(-1)^1 = 3 - 3 = \boxed{0}$

$S_2 = 3(-1)^0 + 3(-1)^1 + 3(-1)^2 = 3 - 3 + 3 = \boxed{3}$

$S_3 = 3(-1)^0 + 3(-1)^1 + 3(-1)^2 + 3(-1)^3 = 3 - 3 + 3 - 3 = \boxed{0}$

(b) State whether or not the series converges and explain how you are determining this.

The series does not converge since $\lim_{n\to\infty} S_n$ dne (sequence has undamped oscillations)

5. (4 pts) Given the series $\sum_{k=0}^{\infty} \frac{2}{k+3} - \frac{2}{k+4}$

(a) Find the nth partial sum of the series. Show work!

$$S_n = \frac{2}{3} \underbrace{- \frac{2}{4} + \frac{2}{4}}_{0} \underbrace{- \frac{2}{5} + \frac{2}{5}}_{0} \underbrace{- \frac{2}{6} + \cdots + \frac{2}{n+3}}_{0} - \frac{2}{n+4}$$

$$\Rightarrow S_n = \frac{2}{3} - \frac{2}{n+4}$$

(b) Use your work in (a) to determine whether the series converges and if so, find the sum of the series.

Since $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2}{3} - \frac{2}{n+4} = \frac{2}{3}$

the series converges. The sum is $\frac{2}{3}$.

6. (6 pts) Write each of the following series in $\sum$ notation, and whether the series converges or diverges. You do not have to justify how you know it converges or diverges.

(a) $1 - \frac{2}{3} + \frac{3}{9} - \frac{4}{27} + \cdots = \sum_{k=0}^{\infty} \frac{(k+1)(-1)^k}{3^k}$ converges diverges

(answers may vary)

also $\sum_{k=1}^{\infty} \frac{k}{3^{k-1}} (-1)^{k+1}$

(b) $\frac{1}{1\cdot 3} + \frac{4}{3\cdot 5} + \frac{9}{5\cdot 7} + \frac{16}{7\cdot 9} + \cdots = \sum_{k=0}^{\infty} \frac{(k+1)^2}{(2k+1)(2k+3)}$ converges diverges

(answers may vary)

also $\sum_{k=1}^{\infty} \frac{k^2}{(2k-1)(2k+1)}$

7. (6 pts) Find the sum of the series $\sum_{k=0}^{\infty} \frac{3^k+5}{4^k}$ $= \sum_{K=0}^{\infty} \left(\frac{3}{4}\right)^K + 5\left(\frac{1}{4}\right)^K$

$$= \sum_{K=0}^{\infty} \left(\frac{3}{4}\right)^K + \sum_{K=0}^{\infty} 5\left(\frac{1}{4}\right)^K$$

$$= \frac{1}{1-\frac{3}{4}} + \frac{5}{1-\frac{1}{4}}$$

$$= \frac{1}{\frac{1}{4}} + \frac{5}{\frac{3}{4}} = 4 + \frac{20}{3} = \boxed{\frac{32}{3}}$$

8. (4 pts) Use the Integral Test to <u>prove</u> that the series either converges or diverges. Use proper notation and clearly state your conclusion!

$\sum_{k=2}^{\infty} \frac{1}{k\ln(k)}$ $\quad f(x) = \frac{1}{x\ln(x)} \Longrightarrow \int_2^{\infty} \frac{1}{x\ln x}dx = \lim_{b\to\infty} \int_2^b \frac{1}{x}\,\frac{1}{\ln x}\,dx$

Diverges by the Integral Test

$$= \lim_{b\to\infty} \ln(\ln(x))\Big|_2^b$$

$$= \lim_{b\to\infty} \ln(\ln(b)) - \ln(\ln(2))$$

$$= \infty \quad \therefore \text{ integral diverges}$$

9. (3 pts) Consider the series $\sum_{k=1}^{\infty} \frac{k^2}{k^3+1}$. Does the series converge or diverge? diverges

Would you be able to use Direct Comparison to prove this? Why or why not?

No, since $\frac{K^2}{K^3+1} < \frac{K^2}{K^3} = \frac{1}{K}$, we can't use Direct Comparison, since $\sum \frac{1}{K}$ diverges. We need a divergent series with terms which are less than the given series.

10. (4 pts) Use a Comparison Test to <u>prove</u> that the series converges or diverges. Clearly state your conclusion!

$$\sum_{k=1}^{\infty} \frac{k^2-1}{k^5+5k^2}$$ behaves like $\sum \frac{k^2}{k^5} = \sum \frac{1}{k^3}$ so suspect convergence.

Direct Comparison:

We know $k^5 < k^5 + 5k^2$

So $\Rightarrow \frac{1}{k^5+5k^2} < \frac{1}{k^5}$

Note: You could also do Limit Comparison

So $\frac{k^2-1}{k^5+5k^2} < \frac{k^2}{k^5+5k^2} < \frac{k^2}{k^5} = \frac{1}{k^3}$

$\therefore$ since $\frac{k^2-1}{k^5+5k^2} < \frac{1}{k^3}$

The series converges by the Direct Comparison Test.

11. (8 pts) Use any test to ~~determine~~ prove convergence or divergence of the series. Clearly state your conclusion.

3 (a) $\sum_{k=1}^{\infty} \frac{5k^3}{k^3+10k}$

$$\lim_{k\to\infty} \frac{5k^3}{k^3+10k} = 5 \neq 0$$

$\therefore$ the series <u>diverges</u>, by the Divergence Test.

5 (b) $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$

Factorials so apply the Ratio Test

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\frac{(k+1)!(k+1)!}{(2k+2)!}}{\frac{k!k!}{(2k)!}} \right|$$

$$= \lim_{k\to\infty} \frac{(k+1)(k+1)\cdot \cancel{k!k!}}{(2k+2)(2k+1)\cancel{(2k)!}} \cdot \frac{\cancel{(2k)!}}{\cancel{k!k!}} = \frac{1}{4} < 1$$

By the Ratio Test, since $\lim \left| \frac{a_{k+1}}{a_k} \right| = \frac{1}{4} < 1$

the series <u>converges</u>.

12. (8 pts) Use any appropriate test(s) to determine and prove the series is absolutely convergent, conditionally convergent, or divergent. Clearly state your conclusion!

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\cos\left(\frac{1}{k}\right)}{k^2}$$

Absolute convergence: Consider $\sum |a_K| = \sum \left| \frac{(-1)^{K+1}\cos\left(\frac{1}{K}\right)}{K^2} \right|$

$$= \sum \frac{\cos\left(\frac{1}{K}\right)}{K^2} *$$

since $\frac{\cos\left(\frac{1}{K}\right)}{K^2} \le \frac{1}{K^2}$, by Direct Comparison we have that the series converges, $\therefore$ the original series converges absolutely.

$*$ $\cos\left(\frac{1}{K}\right)$ is positive for $K = 1, 2, 3, \cdots$ (ask me about this)

13. (6 pts) Given the following power series, determine the *open* interval of convergence. (You do not have to test the endpoints of the interval). Show work!

$$\sum_{k=1}^{\infty} \frac{k(x-1)^k}{6^k}$$

$\lim\limits_{K \to \infty} \left| \frac{a_{K+1}}{a_K} \right| < 1$ guarantees convergence

$$\Rightarrow \lim_{K \to \infty} \left| \frac{\frac{(K+1)(x-1)^{K+1}}{6^{K+1}}}{\frac{K(x-1)^K}{6^K}} \right| = |x-1| \lim_{K \to \infty} \frac{(K+1)}{6K} < 1$$

$$\Rightarrow |x-1| \cdot \frac{1}{6} < 1$$

$$\Rightarrow |x-1| < 6$$

$$\Rightarrow -6 < x - 1 < 6$$

$$-5 < x < 7$$

Interval of convergence = $-5 < x < 7$

14. (10 pts) Use the MacLaurin series for $\frac{1}{1-x}$ to determine the MacLaurin Series for $\frac{1}{1-4x}$

Give the first 4 terms of the series then write the series using $\sum$ notation.
Determine the interval of convergence. Either show work or explain your reasoning for the interval.

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

Interval of Convergence
$|x| < 1$

$$\therefore \quad \frac{1}{1-4x} = 1 + (4x) + (4x)^2 + \cdots$$
$$= \sum_{k=0}^{\infty} 4^k x^k$$

Interval of convergence
$|4x| < 1$
$\Rightarrow |x| < \frac{1}{4}$
OR $-\frac{1}{4} < x < \frac{1}{4}$

15. (10 pts) Using the Taylor Polynomial formula, derive the Taylor polynomial of degree 3 about $x = 1$ for the function $f(x) = \ln(x)$

$f(x) = \ln x$
$f'(x) = \frac{1}{x} = x^{-1}$
$f''(x) = -x^{-2}$
$f'''(x) = 2x^{-3}$

$f(1) = \ln(1) = 0$
$f'(1) = \frac{1}{1} = 1$
$f''(1) = -1$
$f'''(1) = 2$

$f(x) \approx P_3(x)$ where

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$P_3(x) = 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$\boxed{P_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}}$$

16. (5 pts) Use the 3rd degree Taylor polynomial for $\sqrt{1+x}$ to approximate $\sqrt{1.02}$

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

$$\sqrt{1+.02} \approx 1 + \frac{.02}{2} - \frac{(.02)^2}{8} + \frac{(.02)^3}{16}$$

$$\sqrt{1.02} \approx 1.0099505$$