Math 265B: Test 4 Name: KEY

Please show your work in a clear, logical fashion. Credit is based on the correct work not just the final answer. Simplify all answers as much as possible unless otherwise indicated and give exact answers unless an approximation is asked for in the problem. *Only scientific calculators may be used on this exam.*

1. (21 pts) Fill in and/ or circle the correct answer (no need for show work or justify your answer):

(a) $\sum_{k=1}^{\infty} \frac{1}{k^{1.0001}}$ CONVERGES DIVERGES IMPOSSIBLE TO TELL

(b) $\sum_{k=0}^{\infty} \left(\frac{1}{e}\right)^k$ CONVERGES DIVERGES IMPOSSIBLE TO TELL

(c) The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the Harmonic Series and it converges / diverges (circle one).

(d) Does the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converge absolutely? Yes No Impossible to tell

Does $\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{3k^2+1}$ converge conditionally? Yes No Impossible to tell

(e) If $\sum_{k=0}^{\infty} a_k$ diverges, then the series $\sum_{k=1000}^{\infty} a_k$, created by removing 1000 terms, will

CONVERGE DIVERGE IMPOSSIBLE TO TELL

(f) If $a_k \geq b_k$ and $\sum_{k=0}^{\infty} a_k$ diverges, then we know that the series $\sum_{k=0}^{\infty} b_k$ will

CONVERGE DIVERGE IMPOSSIBLE TO TELL

2. (8 pts) Find the value of the infinite series.

$$\sum_{k=0}^{\infty} 2\left(\tfrac{1}{5}\right)^k + \left(-\tfrac{2}{3}\right)^k$$

$$= 2\sum_{k=0}^{\infty}\left(\tfrac{1}{5}\right)^k + \sum_{k=0}^{\infty}\left(-\tfrac{2}{3}\right)^k$$

$$= 2\left(\frac{1}{1-\frac{1}{5}}\right) + \frac{1}{1-\left(-\frac{2}{3}\right)}$$

$$= 2\left(\tfrac{5}{4}\right) + \tfrac{3}{5} = \tfrac{31}{10} = 3.1$$

3. (10 pts) Use the Integral Test to prove that the series either converges or diverges. Use proper notation.

$$\sum_{k=1}^{\infty} ke^{-k^2}$$ This series CONVERGES DIVERGES

$\ast$ $f(x) = xe^{-x^2}$ is positive $x \geq 1$ decreasing and continuous.

Proof: By the Integral Test, we know that if $\int_1^{\infty} xe^{-x^2}\,dx$ $\ast$ converges, then $\sum_{k=1}^{\infty} ke^{-k^2}$ also converges.

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty}\left[\int_1^{b} xe^{-x^2}\,dx\right]^{\ast}$$

$$= \lim_{b\to\infty}\left[-\tfrac{1}{2}e^{-b^2} + \tfrac{1}{2}e^{-1}\right]$$

$\ast$
$$\int_1^{b} xe^{-x^2}\,dx = -\tfrac{1}{2}e^{-x^2}\Big|_1^{b} = -\tfrac{1}{2}e^{-b^2} + \tfrac{1}{2}e^{-1}$$

$$= 0 + \tfrac{1}{2}e^{-1}$$

$\therefore$ the integral converges

$\therefore$ the series converges.

Typo!

4. (8 pts) Use a Comparison Test (either Direct or Limit) to prove that the series converges or diverges. Use proper notation!

$$\sum_{k=2}^{\infty} \frac{k^2}{k^3-1}.$$ This series CONVERGES **DIVERGES**

can't start at $k=1$, division by zero.

Proof:

Direct Comparison:

$$k^3 > k^3 - 1$$

$$\Rightarrow \frac{1}{k^3} < \frac{1}{k^3-1}$$

$$\Rightarrow \frac{k^2}{k^3} < \frac{k^2}{k^3-1}$$

Since $\sum_{k=2}^{\infty} \frac{k^2}{k^3} = \sum_{k=2}^{\infty} \frac{1}{k}$ and this series diverges (p-Series Test, $p=1$), by Direct Comparison $\sum_{k=2} \frac{k^2}{k^3-1}$ also diverges.

Limit Comparison.

Comparing our series to $\sum_{k=2}^{\infty} \frac{1}{k}$, we have

$$\lim_{k\to\infty} \frac{1/k}{k^2/k^3-1} = \lim_{k\to\infty} \frac{k^3-1}{k^3} = 1$$

Since the limit exists the series behave the same and thus both diverge. ($\sum_{k=2}^{\infty} \frac{1}{k}$ diverges by P-series Test.)

5. (6 pts) Use any test to determine convergence or divergence of the series.

$$\sum_{k=1}^{\infty} 2^{\frac{1}{k}}$$ This series CONVERGES **DIVERGES**

Proof:

Divergence Test.

$$\lim_{k\to\infty} 2^{\frac{1}{k}} = 2^0 = 1 \neq 0$$

$\therefore$ the series diverges.

6. (8 pts) Use any appropriate test(s) to determine and prove the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{k=2}^{\infty} \frac{(-1)^k k}{k^2-1}$$

This series CONVERGES ABSOLUTLEY CONVERGES CONDITIONALLY DIVERGES

Proof: To show conditional convergence, first note that the series $\sum_{K=2}^{\infty} |a_K| = \sum_{K=2}^{\infty} \frac{K}{K^2-1}$ is divergent.* So the series is not absolutely convergent.

Second, observe that $\lim_{K\to\infty} \frac{K}{K^2-1} = 0$ $\therefore$ by the A.S.T. the alternating series converges.*

$\therefore$ the series converges conditionally.

*Direct Comparison $\frac{K}{K^2-1} > \frac{1}{K}$ and $\sum_{K=2}^{\infty} \frac{1}{K}$ diverges.

7. (12 pts) Use the Taylor Series formula to find the first 3 non-zero terms for the function $f(x) = \ln(x)$, expanded about x = 1. Then write the series using $\sum$ notation

*Note: We're not proving $a_{K+1} < a_K$ but if we did, we'd have to show that $\frac{K+1}{(K+1)^2-1} < \frac{K}{K^2-1}$. Here's the proof of that:

$$\frac{K+1}{(K+1)^2-1} = \frac{K+1}{K^2+2K} < \frac{K+1}{K^2+K} = \frac{K+1}{K(K+1)} = \frac{1}{K}$$

$$\frac{1}{K} = \frac{K}{K^2} < \frac{K}{K^2-1} \quad \therefore \frac{K+1}{(K+1)^2-1} < \frac{K}{K^2-1}$$

$\therefore a_{K+1} < a_K$

$f(x) = \ln x$	$f(1) = 0$
$f'(x) = \frac{1}{x} = x^{-1}$	$f'(1) = 1$
$f''(x) = -1x^{-2}$	$f''(1) = -1$
$f'''(x) = 2\cdot 1 x^{-3}$	$f'''(1) = 2$

Series: $0 + 1(x-1)^1 - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \cdots$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots$$

$$= \sum_{K=1}^{\infty} \frac{(-1)^{K+1}(x-1)^K}{K}$$

Answers may vary

8. (16 pts) Given the following power series, determine the *open* interval of convergence. (You do not have to test the endpoints of the interval). Show work!

(a) $\sum_{k=1}^{\infty} \frac{5^k x^k}{k!}$

We will force convergence by applying the Ratio Test:

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{\frac{5^{k+1} x^{k+1}}{(k+1)!}}{\frac{5^k x^k}{k!}} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{5^k \cdot 5^1 \cdot x^k \cdot x^1}{(k+1)\cdot k!} \cdot \frac{k!}{5^k \cdot x^k} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{5x}{k+1} \right| < 1$$

$$5|x| \cdot \underbrace{\lim_{k\to\infty} \frac{1}{k+1}} < 1$$

$$5|x| \cdot 0 < 1$$

This is true for <u>all</u> possible values for x so the interval is all real numbers.

(Note: The <u>radius</u> of convergence is $R=\infty$)

Interval of convergence = $(-\infty, \infty)$

(b) $\sum_{k=1}^{\infty} \frac{(x-1)^k}{k^2}$

$$\lim_{k\to\infty} \left| \frac{\frac{(x-1)^{k+1}}{(k+1)^2}}{\frac{(x-1)^k}{k^2}} \right| < 1$$

$$\lim_{k\to\infty} \left| \frac{\cancel{(x-1)^k}(x-1)^1}{k^2+2k+1} \cdot \frac{k^2}{\cancel{(x-1)^k}} \right| < 1$$

$$|x-1| \cdot \underbrace{\lim_{k\to\infty} \frac{k^2}{k^2+2k+1}} < 1$$

$$|x-1| \cdot 1 < 1 \,\star$$

$$-1 < x-1 < 1$$

$$0 < x < 2$$

$\star$ Note: The <u>radius</u> of convergence is $R=1$

Interval of convergence = $(0,2)$

9. (5 pts) Use the given series for cos(x) to find the series for $2x\cos(x^2)$

Write the first 4 terms of the series then write the series using $\sum$ notation

Given: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

Find: $2x\cos(x^2) =$

$$= 2x\left[1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots\right]$$

$$\boxed{= 2x - \frac{2x^5}{2!} + \frac{2x^9}{4!} - \frac{2x^{13}}{6!} + \cdots} = \boxed{\sum_{K=0}^{\infty} \frac{2(-1)^K x^{4K+1}}{(2K)!}}$$

1^{st} 4 terms $\qquad \sum$ notation

10. (6 pts) Differentiate the series $x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$ then write the derivative series in $\sum$ notation

$$\frac{d}{dx}\left[x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right]$$

$$= 2x - \frac{6x^5}{3!} + \frac{10x^9}{5!} - \frac{14x^{13}}{7!} + \cdots$$

$$= \sum_{K=0}^{\infty} \frac{(-1)^K (4K+2) x^{4K+1}}{(2K+1)!}$$

(OR Teagan's & Gabe's clever form - see if you can show the 2 are equivalent!)

$$= \sum_{K=0}^{\infty} \frac{(-1)^K 2x^{4K+1}}{(2K)!}$$

Simpler, hence better, form!

MacLaurin Series for select functions:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Extra credit (4 pts): Derive Euler's Formula using series for sin(x), cos(x), and e^x

$e^{i\theta} = \cos\theta + i\sin\theta$ By direction substitution, we get

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots$$

But $i^2 = -1$ $\quad i^3 = i^2 \cdot i = -i$, $\quad i^4 = i^2 \cdot i^2 = 1$

$i^5 = i^4 \cdot i = i$, $\quad i^6 = i^2 \cdot i^4 = -1$, $\quad i^7 = i^4 \cdot i^3 = -i$

and thus the pattern repeats itself.

(In general, $i^n = i^{4K+R} = (i^4)^k \cdot i^R = 1^k i^R = i^R$)

$$\text{so } e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots$$

Separating these terms into real and imaginary parts,

$$e^{i\theta} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots}_{\text{Real}} + \underbrace{i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + \cdots}_{\text{Imaginary}}$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$= \cos\theta + i\sin\theta$ ▨ Q.E.D.