Test 4 Take Home Key

1) $\sum_{k=0}^{\infty}\left(\frac{x-1}{3}\right)^k$ — Since this is a geometric series, we get convergence for $|r|<1$, where $r=\frac{x-1}{3}$ (the base)

so $\left|\frac{x-1}{3}\right|<1$

$$\Rightarrow -1<\frac{x-1}{3}<1$$

$$\Rightarrow -3<x-1<3$$

$$\Rightarrow -2<x<4$$

$\therefore$ the interval of convergence is $-2<x<4$

2. (a) Given. $\sum_{k=1}^{\infty}\frac{(-2)^k x^k}{k}$

we get convergence by the Ratio Test, if

$$\lim_{k\to\infty}\left|\frac{\frac{(-2)^{k+1}x^{k+1}}{k+1}}{\frac{(-2)^k x^k}{k}}\right|<1$$

$$\Rightarrow \lim_{k\to\infty}\left|\frac{(-2)x\cdot k}{k+1}\right|<1$$

$$\Rightarrow 2|x|\lim_{k\to\infty}\frac{k}{k+1}<1$$

$$\Rightarrow 2|x|\cdot 1<1$$

$\Rightarrow |x|<\frac{1}{2}$ ← Radius of convergence

Interval of convergence: $-\frac{1}{2}<x<\frac{1}{2}$

2(a) continued (<u>optional</u>)

at the endpoints we have the following:

For $x = -\frac{1}{2} \Rightarrow \sum_{k=1}^{\infty} \frac{(-2)^k x^k}{k} = \sum_{k=1}^{\infty} \frac{(-2)^k (-\frac{1}{2})^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$

this series diverges (harmonic / p-Test)

For $x = \frac{1}{2} \Rightarrow \sum_{k=1}^{\infty} \frac{(-2)^k (\frac{1}{2})^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

this series converges (Alternating Series Test)

so the final interval of convergence is

$-\frac{1}{2} < x \leq \frac{1}{2}$ or $\left(-\frac{1}{2}, \frac{1}{2}\right]$

2 (b) Given $\sum_{K=1}^{\infty} \frac{K^2}{K!} x^K$

We get convergence for

$$\lim_{K\to\infty} \left| \frac{\frac{(K+1)^2 x^{K+1}}{(K+1)!}}{\frac{K^2 x^K}{K!}} \right| < 1$$

$$\lim_{K\to\infty} \left| \frac{(K+1)^{\cancel{2}}\, x \cdot \cancel{K!}}{K^2\, \cancel{(K+1)}\, \cancel{K!}} \right| < 1$$

$$|x| \lim_{K\to\infty} \frac{K+1}{K^2} < 1$$

$$\underbrace{|x| \cdot 0 < 1}$$

This is true for x = any real number

∴ the interval of convergence is $(-\infty, \infty)$
which makes the radius of convergence ∞.

3(a) $f(x) = e^{x^2}$; derive Taylor series about $x = 0$

$f(x) = e^{x^2}$ | $f(0) = e^0 = 1$

$f'(x) = 2xe^{x^2}$ | $f'(0) = 2 \cdot 0 e^{0^2} = 0$

$f''(x) = 2e^{x^2} + 4x^2 e^{x^2}$ | $f''(0) = 2e^0 + 4(0)e^0 = 2$

$$e^{x^2} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

$$= 1 + 0x + \frac{2x^2}{2!} + \cdots$$

$$= 1 + x^2 + \cdots$$

(b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

4. $\sqrt{1+x}$, find $p_3(x)$ about $x=0$, approx $\sqrt{.96}$

$f(x) = (1+x)^{1/2}$ $\quad f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$ $\quad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{2\cdot 2}(1+x)^{-\frac{3}{2}}$ $\quad f''(0) = -\frac{1}{4}$

$f'''(x) = +\frac{1\cdot 3}{2\cdot 2\cdot 2}(1+x)^{-\frac{3}{2}}$ $\quad f'''(0) = +\frac{3}{8}$

$p_3(x) = 1 + \frac{1}{2}x - \frac{\frac{1}{4}}{2!}x^2 + \frac{\frac{3}{8}}{3!}x^3$

$\implies p_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

$\sqrt{.96} = \sqrt{1+(-.04)} \approx p_3(-.04)$

$= 1 + \frac{1}{2}(-.04) - \frac{1}{8}(-.04)^2 + \frac{1}{16}(-.04)^3$

$= .979796$

ABSOLUTE ERROR: $|R(x)| = |f(x) - p_3(x)|$

calculator: $\sqrt{0.96} = .9797958971 = f(-.04)$

$|R(-.04)| = |f(-.04) - p_3(-.04)|$

$= |.9797958971 - .979796|$

$\cong 1.03 \times 10^{-7}$

5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$

Interval of convergence:

$-1 < x \leq 1$

$$x^2 \ln(1+4x) = x^2\left[(4x) - \frac{(4x)^2}{2} + \frac{(4x)^3}{3} - \frac{(4x)^4}{4} + \cdots\right]$$

$$= 4x^3 - \frac{4^2 x^4}{2} + \frac{4^3 x^5}{3} - \frac{4^4 x^6}{4} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 4^k x^{k+2}}{k}$$

Interval of Convergence:

$-1 < 4x \leq 1$

$-1/4 < x \leq 1/4$

6. $$\frac{e^x - e^{-x}}{x} = \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)}{x}$$

$$= \frac{2x + \frac{2x^3}{3!} + \cdots}{x}$$

$$= 2 + \frac{2x^2}{3!} + \cdots$$

$$\lim_{x \to \infty} \frac{e^x - e^{-x}}{x} = \lim_{x \to \infty} 2 + \frac{2x^3}{3!} + \cdots = \boxed{2}$$

7. $$\frac{\sin(x^2)}{x^2} = \left[(x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right] / x^2$$

$$= 1 + \frac{x^4}{6} + \frac{x^8}{120} + \cdots$$

$$\int_1^2 \frac{\sin(x^2)}{x^2}\, dx \cong \int_1^2 1 + \frac{x^4}{6} + \frac{x^8}{120}\, dx$$

$$= x - \frac{x^5}{30} + \frac{x^9}{1080} \Big|_1^2$$

$$= \left(2 - \frac{2^5}{30} + \frac{2^9}{1080}\right) - \left(1 - \frac{1}{30} + \frac{1}{1080}\right)$$

$$\cong \frac{95}{216} \cong .4398$$

8, Derive $e^{i\theta} = \cos\theta + i\sin\theta$ using Series:

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots$$

Note:
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + \left(i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \cdots\right)$$

Real part   Imaginary part

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta \qquad + i\sin\theta$$

Q.E.D